Introduction to perverse schobers

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Summary: ToDo

Prerequisites: We will assume basic knowledge of ∞-category theory but recall some relevant notions in the first lecture. We assume familiarity with the notion of a spherical functor.

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1 Lecture 1: Lax higher category theory

1.1 Stable ∞-categories and lax limits

The theory of ∞-categories can be developed using quasi-categories, meaning simplicial sets admitting inner horn fillers. Quasi-categories were studied by Joyal and Lurie, see [Lur09, Lur17], as well as [Lur, Cis19] for more recent accounts.

We will assume prior exposure to the theory of ∞-category, but recall some of the relevant notions in this lecture. Given an ∞-category $\mathcal{C}$ and two objects $X, Y \in \mathcal{C}$, we write $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ for the space of morphisms from X to Y in $\mathcal{C}$. Given an ∞-category $\mathcal{C}$, its homotopy category $\operatorname{Ho}\mathcal{C}$ is the 1-category with the same objects and Hom sets defined as $\operatorname{Hom}_{\operatorname{Ho}\mathcal{C}}(X, Y) = \pi_0 \operatorname{Hom}_{\mathcal{C}}(X, Y)$. We will shortly introduce stable ∞-categories. The homotopy 1-categories of stable ∞-categories inherit a triangulated structure and virtually all triangulated categories of interest admit such an ∞-categorical *enhancement*.

Stable ∞-categories

Recall that an object X in an ∞-category $\mathcal{C}$ is called initial if $\operatorname{Hom}_{\mathcal{C}}(X, Y) \simeq *$ for all $Y \in \mathcal{C}$. A terminal object is an initial object in $\mathcal{C}^{\mathrm{op}}$. A zero object $0 \in \mathcal{C}$ is an object that is both initial and terminal. An ∞-category is called pointed if it admits a zero object.

Recall further that a *fiber sequence* in a pointed ∞-category in $\mathcal{C}$ refers to a pullback square of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "f"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

In this case, we call X the fiber of f and write $X = \mathrm{fib}(f)$.

Note that as usual in ∞-categories, a square as above only commutes up to homotopy and this homtopy from $X \to Y \to Z$ to $X \to 0 \to Z$ is part of the data contained in the square. In the homotopy category $\mathrm{Ho}\,\mathcal{C}$, the square becomes strictly commutative. The square being pullback means that the space of morphisms to X is equivalent to the space of cones over the diagram [Lur, Tag 02VZ].

Dually, a *cofiber sequence* in a pointed ∞-category refers to a pushout square of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

In this case, we call Z the cofiber of f and write $Z = \mathrm{cof}(f)$.

Definition 1.1. An ∞-category $\mathcal{C}$ is called stable if

- it admits a zero object $0 \in \mathcal{C}$,
- every morphism $f\colon x \to y$ in $\mathcal{C}$ admits a fiber and a cofiber,
- any square in $\mathcal{C}$ of the form

  $$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

  is a fiber sequence if and only if it is a cofiber sequence.

Theorem 1.2 ([Lur17, Thm. 1.1.2.14])**.** *If $\mathcal{C}$ is stable, its homotopy category* $\mathrm{Ho}\,\mathcal{C}$ *inherits the structure of a triangulated category.*
A sequence $X \to Y \to Z \to X[1]$ in $\mathrm{Ho}\,\mathcal{C}$ *forms a distinguished triangle if there are pullback and pushout squares in $\mathcal{C}$ as follows:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

A functor between stable ∞-categories is called exact if it preserves fiber and cofiber sequences.

The theory of stable ∞-categories resolves several structural issues with the theory of triangulated categories. For instance, the cone, i.e. cofiber, is functorial. Further, given two stable ∞-categories $\mathcal{C}, \mathcal{D}$, there is a stable ∞-category $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ of exact functors.

A further reason to consider stable ∞-categories over triangulated categories is that the collection of all stable ∞-categories itself form a well behaved ∞-category. We denote by St this ∞-category of (small) stable ∞-categories and exact functors. We can consider limits and colimits in the ∞-category St, which allows in various ways to construct new stable ∞-categories from given ones.

Lax limits

The ∞-category St of stable ∞-categories admits a natural $(\infty, 2)$-categorical structure, we denote this $(\infty, 2)$-category by $\mathbf{St}$. This means that its Hom spaces can be upgraded to ∞-categories (namely the functor categories). Let Z be a (small) ∞-category and fix a functor $F\colon Z \to \mathrm{St}$, meaning a homotopy commutative diagram of stable ∞-categories and exact functors. Then F admits a limit and colimit. There is a further notion of a *lax limit* of the diagram Z in $\mathbf{St}$. A *lax cone* over F consists, roughly speaking, of a stable ∞-category $\mathcal{C}$ and a collection of exact functors $p_z\colon \mathcal{C} \to F(z)$ for all $z \in Z$, together with a natural transformation

lax cone:
$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "p_z"'] \arrow[dr, "p_{z'}"] \arrow[d, phantom, "\Nearrow"] & \\ F(z) \arrow[rr, "F(\alpha)"'] & {} & F(z') \end{tikzcd}$$

for each morphism $\alpha\colon z \to z'$ in Z. Note that the natural transformation is not required to be a natural equivalence. The *lax limit* $\lim^{\mathrm{lax}}(F) \in \mathrm{St}$ of F is the universal stable ∞-category equipped with a lax cone over F, meaning that the ∞-category of functors $\mathcal{C} \to \lim^{\mathrm{lax}}(F)$ is equivalent to the ∞-category of lax cones over F with tip $\mathcal{C}$.

There are similar notions of *oplax limit*, *lax colimit* and *oplax colimit* of a functor $F\colon Z \to \mathrm{St}$. The oplax limit is equipped with a universal oplax cone, meaning roughly a cone as follows:

oplax cone:
$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "p_z"'] \arrow[dr, "p_{z'}"] \arrow[d, phantom, "\Swarrow"] & \\ F(z) \arrow[rr, "F(f)"'] & {} & F(z') \end{tikzcd}$$

Lax and oplax colimit cocones are in turn of the following form:

lax cocone:
$$\begin{tikzcd} F(z) \arrow[rr, "F(\alpha)"] \arrow[dr, "i_z"'] & {} \arrow[d, phantom, "\Nearrow"] & F(z) \arrow[dl, "i_{z'}"] \\ & \mathcal{C} & \end{tikzcd}$$

oplax cocone:
$$\begin{tikzcd} F(z) \arrow[rr, "F(\alpha)"] \arrow[dr, "i_z"'] & {} \arrow[d, phantom, "\Swarrow"] & F(z') \arrow[dl, "i_{z'}"] \\ & \mathcal{C} & \end{tikzcd}$$

Due to space and time constraints, we can only give indications of the $(\infty, 2)$-categorical notions. For the interested reader, we remark that a lax cone can be defined as a lax natural transformation from the constant diagram, see for instance [AHM26, Not. 2.1.9].

Example 1.3. Consider the poset category $[2] = \{1 \to 2\}$. We note for the simplicially-minded reader that that $[2] \simeq \Delta^1$.

A functor $F\colon [2] \to \mathrm{St}$ amounts to an exact functor

$$F \coloneqq F(1 \to 2)\colon \mathcal{C} \coloneqq D(1) \to \mathcal{D} \coloneqq D(2)$$

between stable ∞-categories (note the abusive notation). The limit of F is given by $\mathcal{C}$ and the colimit of F is given by $\mathcal{D}$. Let us describe the lax limit of D.

The forgetful functor $\mathrm{St} \to \mathrm{Cat}_\infty$ preserves lax limits. Objects of $\lim^{\mathrm{lax}}(F)$ can be identified with functors $* \to \lim^{\mathrm{lax}}(F)$, which by the universal property of the lax limit cone in Cat_∞ amount to lax cones:

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Nearrow"] & \\ \mathcal{C} \arrow[rr, "F"'] & {} & \mathcal{D} \end{tikzcd}$$

Such a lax cones amounts to choosing two objects $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and a morphism $\eta\colon F(X) \to Y$ in $\mathcal{D}$. We thus write elements of $\lim^{\mathrm{lax}}(D)$ as triples

$$(X, Y, \eta\colon F(X) \to Y) \in \lim^{\mathrm{lax}}(F)\,.$$

The functors in the lax limit cone

$$\begin{tikzcd} & \lim^{\mathrm{lax}}(F) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{C} \arrow[rr, "F"'] \arrow[rr, phantom, shift left=4, "\Rightarrow"] & & \mathcal{D} \end{tikzcd}$$

are given on objects by

$$p_1(X, Y, \eta) = X$$
$$p_2(X, Y, \eta) = Y$$

and the natural transformation $F \circ p_1 \to p_2$ evaluates at (X, Y, η) to η.

The functors p_1, p_2 admit adjoints $o_1 \dashv p_1 \dashv q_1$ and $m_2 \dashv o_2 \dashv p_2$, whose action on objects is as follows:

$$\begin{aligned} o_1\colon \mathcal{C} \to \lim^{\mathrm{lax}}(F), &\quad X \mapsto (X, F(X), \mathrm{id}_{F(x)}) \\ q_1\colon \mathcal{C} \to \lim^{\mathrm{lax}}(F), &\quad X \mapsto (X, 0, 0) \\ m_2\colon \lim^{\mathrm{lax}}(F) \to \mathcal{D}, &\quad (X, Y, \eta) \mapsto \mathrm{cof}(\eta) \\ o_2\colon \mathcal{D} \to \lim^{\mathrm{lax}}(F), &\quad Y \mapsto (0, Y, 0)\,. \end{aligned}$$

These functors assemble into a recollement:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "o_2" description] & \lim^{\mathrm{lax}}(F) \arrow[l, shift right=3, "m_2"'] \arrow[l, shift left=3, "p_2"] \arrow[r, "p_1" description] & \mathcal{C} \arrow[l, shift right=3, "o_1"'] \arrow[l, shift left=3, "q_1"] \end{tikzcd}$$

The *gluing functor* of such a recollement is defined as the composite $p_2 \circ o_1$, but note that $p_2 \circ o_1 \simeq F$. In fact, any recollement of stable ∞-categories is equivalent to the lax limit of its gluing functor, see [DKSS24, Prop. 2.2.11].

Theorem 1.4 (Lax additivity, [CDW26])**.** *(i) Let $F\colon [2] \to \mathbf{St}$ be a functor. Then*

$$\lim^{\mathrm{lax}}(F) \simeq \lim^{\mathrm{oplax}}(F) \simeq \mathrm{colim}^{\mathrm{lax}}(F) \simeq \mathrm{colim}^{\mathrm{oplax}}(F)$$

We also write

$$\mathcal{C} \oplus_F^{\rightarrow} \mathcal{C} := \lim^{\mathrm{lax}}(F)$$

for the lax limit of $F\colon \mathcal{C} \to \mathcal{D}$, calling it the lax sum.

(ii) For a general functor $F\colon Z \to \mathbf{St}$, there is a canonical equivalence

$$\lim^{\mathrm{lax}}(F) \simeq \mathrm{colim}^{\mathrm{oplax}}(F)\,.$$

Remark 1.5. Every dg algebra A gives rise to a stable ∞-category $\mathcal{D}^{\mathrm{perf}}(A)$, called the derived ∞-category of C. One can define $\mathcal{D}^{\mathrm{perf}}(A)$ as the dg-nerve of the dg category $\mathrm{Perf}(A)$ of compact and fibrant-cofibrant dg A-modules, see for instance [Chr22, §2.4] for a discussion.

A perfect dg bimodule $M \in \mathrm{dgMod}_B^A$ gives rise to a functor $(\text{-}) \otimes_A^{\mathrm{dg}} M\colon \mathcal{D}^{\mathrm{perf}}(A) \to \mathcal{D}^{\mathrm{perf}}(B)$. The lax limit of $(\text{-}) \otimes_A^{\mathrm{dg}} M$ is described by the *gluing* of A and B along the bimodule M (this is a known dg categorical construction). Concretely, this gluing is given by the upper triangular dg algebra

$$A \oplus_M B = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

whose underlying chain complex is $A \oplus M \oplus B$, and where the multiplication is 'upper-triangular', see also [Chr22, Def. 2.36]. There is an equivalence of stable ∞-categories

$$\mathcal{D}^{\mathrm{perf}}(A \oplus_M B) \simeq \lim^{\mathrm{lax}}(M \otimes_A^{\mathrm{dg}} (\text{-}))$$

see [Chr22, Prop. 2.39].

1.2 Philosophy of categorification

The term *categorified homological algebra*, proposed in [Dyc21], captures the idea of a theory of homological algebra in which vector spaces or abelian groups are replaced with triangulated categories. There are by now a number of results which can be considered as part of this theory, though many fundamental concepts from classical homological algebra remain elusive in the categorified setting. We summarize some of the rules for how to categorify concepts from homological algebra in Table 1.

Classical	Categorified
abelian group A	stable ∞-category $\mathcal{A}$
element $x \in A$	object $X \in \mathcal{A}$
$x - y \in A$	$\mathrm{fib}(x \xrightarrow{f} y) \in \mathcal{A}$
direct sum $A \oplus B$	lax bilimit $\mathcal{A} \oplus_F^{\rightarrow} \mathcal{B}$
additive 1-category	lax additive $(\infty, 2)$-category
abelian category	???
chain complex	categorical chain complex
totalization of a bicomplex	directed categorical totalization
derived category	???

Table 1: The rules of categorification (and its current limits)

To decategorify, starting at a stable ∞-category, one can pass to its Grothendieck group K_0. For instance, the categorification of the difference $x - y$ of two elements of an abelian group is given by the fiber of a morphism $f\colon X \to Y$, which induces the difference in K_0. Some important observations here are:

- Consider two elements x, y with categorifications X, Y. To categorify a difference $x - y$, we must ask for the data of an additional morphism $X \to Y$. Different choices of this morphism lead to different notions of 'difference',i.e. fiber, of X, Y.
- The sum of two elements $x+y$ is a symmetric notion. Asking for an extension $X \to E \to Y$ or $Y \to E' \to X$, so that E, E' are the sums of X, Y introduces a *directedness* to the categorification.

This type of directedness is a recurring feature in this type of categorification. Another instance of this is the categorification of the direct sum of abelian groups $A \oplus B$. Of course, there is the naive categorification $\mathcal{A} \times \mathcal{B}$, i.e. the product category. But more generally, we can ask for the datum of an exact functor $F\colon \mathcal{A} \to \mathcal{B}$ and categorify $A \oplus B$ by the lax sum $A \oplus_F^{\rightarrow} \mathcal{B}$ that is at the same time the lax limit and oplax colimit.

Example 1.6. The abelian category of perverse sheaves on $(\mathbb{C}, 0)$ is equivalent to the abelian category of diagrams of vector spaces and linear maps

$$f\colon \Phi \longleftrightarrow \Psi : g \tag{1}$$

satisfying that $gf - \mathrm{id}_\Phi$ and $fg - \mathrm{id}_\Psi$ are invertible. The vector space Φ is called the *vanishing cycles* of the perverse sheaf and Ψ is called the *nearby cycles*.

According to Table 1, we can categorify this as follows: we replace the vanishing cycles Ψ by a stable ∞-category $\mathcal{V}$ and the nearby cycles Φ by a stable ∞-category $\mathcal{N}$. We further replace the linear maps f, g by exact functor $F\colon \mathcal{V} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{V}$.

To be able to categorify the differences $gf - \mathrm{id}_\Phi$ and $fg - \mathrm{id}_\Psi$, we ask for natural transformation $u\colon \mathrm{id}_\mathcal{V} \to GF$ and $\mathrm{cu}\colon FG \to \mathrm{id}_\mathcal{N}$. It turns out to be a good idea to ask that these natural transformation satisfy the triangle identities and thus exhibit and adjunction $F \dashv G$.

We thus categorify the diagram of vector spaces (1) via an adjunction of stable ∞-categories

$$F\colon \mathcal{V} \longleftrightarrow \mathcal{N} : G$$

satisfying that

- the twist $T_\mathcal{V} \coloneqq \mathrm{cof}(\mathrm{id}_\mathcal{V} \xrightarrow{\text{unit}} GF)\colon \mathcal{V} \to \mathcal{V}$ is an equivalence and
- the cotwist $T_\mathcal{N} \coloneqq \mathrm{fib}(FG \xrightarrow{\text{counit}} FG)\colon \mathcal{N} \to \mathcal{N}$ is an equivalence.

We thus arrive at the notion of a spherical adjunction (F,G are also called spherical functors), in the sense of [AL17].

2 Lectures 2&3: Perverse schobers

Introductory remarks

The word schober is german, referring to hay stacks. A perverse schober is intended to be a perverse sheaf of stable ∞-categories. Their existence was proposed by Kapranov–Schechtman [KS14].

A central issue in the study of perverse schobers is that there is currently no general theory of perverse schobers. With the exception of specific examples of stratified spaces, the existence of a notion of perverse schober remains conjectural. For the existing examples, there is further no general sheaf theory for perverse schobers, allowing for operations such as pushforward or pullback. Spaces for which notions of perverse (pre-)schobers exist include discs and surfaces (see below), symmetric products of discs [DW25], see also [vdK25] for examples, hyperplane arrangements [KS16b, ŠdB22, CDW23], and hyper toric varieties [GH25].

The reader may ask why it is so difficult to define perverse schobers. As there is no proposed definition of the derived category of the ∞-category of stable ∞-categories, we cannot define perverse schobers as the heart of a t-structure on the derived category of constructible sheaves of stable ∞-categories. The next section summarizes the current approach to defining perverse schobers.

2.1 Ad-hoc categorification and pre-schobers vs schobers

Given a stratified space X with locally closed and even dimensional strata, we denote by $\mathrm{Perv}(X)$ the category of topological perverse sheaves on X (with respect to the middle perversity).

Ad-hoc categorification (after Kapranov–Schechtman): Describe $\mathrm{Perv}(X)$ in terms of linear algebra data and categorify according to Table 1.

Example 2.1. A spherical adjunction is an ad-hoc categorification of a perverse sheaf on $\mathbb{C}$ stratified by the origin, described in terms of its nearby and vanishing cycles.

The process of ad-hoc categorification is not straightforward:

- Firstly, for a given stratified space X, finding a description of $\mathrm{Perv}(X)$ in terms of linear algebraic data is non-trivial.
- Even if one has found a description of $\mathrm{Perv}(X)$, identifying the *correct* categorification (as one can typically conceive many different ones) is difficult.

Of course, nothing guarantees that there always exists a unique 'correct' categorification. One way of testing whether an ad-hoc categorification can be 'correct' is by determining whether a given ad-hoc categorification can be translated into an ad-hoc categorification of different descriptions of the category $\mathrm{Perv}(X)$. A good ad-hoc categorification should give rise to a collection of equivalent descriptions of perverse schobers.

Following a proposal of Timothy Logvinenko, we will call the result of a single ad-hoc categorification of a perverse sheaf a *pre-schober*. In line with the previous paragraph, a *perverse schober* is then the result of a collection of results, showing that different notions of pre-schobers on a given stratified space are equivalent. This is a loosely defined notion. We will illustrate this notion in two cases: perverse schobers on $\mathbb{C}$ stratified by 0, and more generally perverse schobers on marked surfaces.

2.2 Warm-up: Perverse schobers on $\mathbb{C}$ via n-spiders

Definition 2.2. The n-spider Sp_n is the ribbon graph with a single n-valent vertex v and n indicent external edges $e_1, \dots, e_n$.

Definition 2.3. A pre-schober on the n-spider Sp_n consists of a collection of functors between stable ∞-categories

$$(F_i\colon \mathcal{V}^n \to \mathcal{N}_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$$

such that

(1) if $n = 1$, the functor F_i is spherical,

(2) if $n \geq 2$ the following conditions hold for all $i \in \mathbb{Z}/n\mathbb{Z}$:

a) The functor F_i admits right adjoints $F_i \dashv G_i \dashv H_i$,

b) the functor G_i is fully faithful, meaning that $F_i \circ G_i \simeq \mathrm{id}_{\mathcal{N}_i}$ via the counit,

c) the composite $F_i \circ G_j\colon \mathcal{N}_j \to \mathcal{N}_i$ vanishes if $j \neq i, i-1$,

d) and $\mathrm{fib}(F_i) = \mathrm{fib}(H_{i-1})$ as full subcategories of $\mathcal{V}^n$.

Exercise 1. Show that condition d) implies that $F_i \circ G_{i-1}\colon \mathcal{N}_{i-1} \to \mathcal{N}_i$ is an equivalence of ∞-categories.

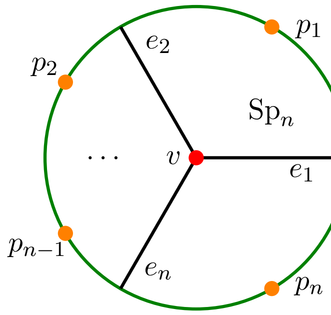
Remark 2.4. Passing to K_0, a pre-schober on the n-spider yields morphisms between abelian groups

$$(f_i\colon \Phi^n \longleftrightarrow \Psi_i : g_i)_{i\in \mathbb{Z}/n\mathbb{Z}}$$

where $\Phi^n = K_0(\mathcal{V}^n)$ and $\Psi_i = K_0(\mathcal{N}_i)$. These satisfy in the case $n \geq 2$ that

- $f_i \circ g_i = \mathrm{id}_{\Psi_i}$,
- $f_i \circ g_{i-1}\colon \Psi_{i-1} \to \Psi_i$ is an isomorphism (by Exercise 1),
- $f_i \circ g_j = 0$ for $j \neq i, i-1$.

Such a collection of abelian groups and group homomorphisms (or the analog of this data over a field) describes a perverse sheaf on $\mathbb{C}$ stratified at the origin, see [KS16a]. Let us fix notation for the n-spider as follows



and let F be the perverse sheaf on $\mathbb{C}$. Then

$$\Phi^n = H^0 R\Gamma_{\mathrm{Sp}_n}(\mathbb{C}, F) \coloneqq H^0 \operatorname{fib}(R\Gamma(\mathbb{C}, F) \to R\Gamma(\mathbb{C}\setminus \mathrm{Sp}_n, F))$$

describes the derived sections with support on the n-spider (which are concentrated in degree 0). Further

$$\Psi_i = H^{-1} F_{p_i}$$

is the stalk of F at p_i (concentrated in degree -1). The morphism $f_i\colon \Psi^n \to \Phi_i$ arises from the connecting homomorphism.

Note that the conditions a)-d) in Definition 2.3 are a non-trivial way to categorify the conditions 1)-3) above.

To describe the relation between pre-schobers on different spiders, we first define the corresponding $(\infty, 2)$-categories of pre-schobers. Denote by $\mathbf{St}$ the $(\infty, 2)$-category of small stable ∞-categories and exact functors. We denote by $\mathbf{Adj}$ the free adjunction [SS86]. An $(\infty, 2)$-functor $\mathbf{Adj} \to \mathbf{St}$ amounts to an adjunction between stable ∞-categories. The $(\infty, 2)$-categorical functor category $\mathbf{Fun}(\mathbf{Adj}, \mathbf{St})$ is called the $(\infty, 2)$-category of adjunctions in $\mathbf{St}$. There is similarly a 2-category $\mathbf{Adj}^n$ such that functors $\mathbf{Adj}^n \to \mathbf{St}$ describe n adjunctions with a common left side.

Definition 2.5. (1) For $n \geq 1$, we denote by $\mathbf{Perv}(\mathrm{Sp}_n) \subset \mathbf{Fun}(\mathbf{Adj}^n, \mathbf{St})$ the full subcategory consisting of n adjunctions $F_i \dashv G_i$, such that the functors $(F_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ define a pre-schober on the n-spider.

(2) In the case $n = 1$, we also write $\mathbf{Sph} = \mathbf{Perv}(\mathrm{Sp}_1)$.

Remark 2.6. We can identify morphisms in $\mathbf{Sph}$ with natural transformations $\eta\colon F \to F'$ that satisfy a Beck-Chevalley condition (i.e. are adjointable), meaning that the corresponding mate is a natural equivalence. One can show that such natural transformation automatically satisfy all iterated adjointability conditions involving the iterated adjoints of F, F'. The same is true for $\mathbf{Perv}(\mathrm{Sp}_n)$.

Theorem 2.7 ([CHQ23, ACJ])**.**

(i) Let

$$\mathcal{F}_n = (F_i \colon \mathcal{V}^n \to \mathcal{N}_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$$

be a pre-schober on the n-spider. Then, for any $j \in \mathbb{Z}/n\mathbb{Z}$, the collection of functors

$$\mathcal{F}^j_{n-1} := \left(F_i|_{\mathrm{fib}(F_j)} \colon \mathrm{fib}(F_j) \to \mathcal{N}_i \right)_{i \in \mathbb{Z}/n\mathbb{Z} \setminus \{j\}}$$

defines a pre-schober on the $(n-1)$-spider.

(ii) The passage to the fiber of F_j defines an equivalence of $(\infty, 2)$-categories

$$\mathbf{Perv}(\mathrm{Sp}_n) \simeq \mathbf{Perv}(\mathrm{Sp}_{n-1}) \, .$$

In particular, for any $n \geq 1$, the $(\infty, 2)$-category $\mathbf{Perv}(\mathrm{Sp}_n)$ is equivalent to the $(\infty, 2)$-category of spherical adjunctions.

Example 2.8. Let $F \colon \mathcal{V} \to \mathcal{N}$ be a spherical functor with right adjoint G. We describe the perverse schober on the 2-spider corresponding to F in the sense of Theorem 2.7.

Let $\mathcal{V}^2_F = \lim^{\mathrm{lax}}(F)$ be the lax limit of the functor $[2] = \{1 \to 2\} \to \mathrm{St}$, $(1 \to 2) \mapsto F$. We can write elements of $\mathcal{V}^2_F$ as triples (x, y, η) with $X \in \mathcal{V}$, $Y \in \mathcal{N}$ and $\eta \colon F(X) \to Y$ a morphism in $\mathcal{N}$.

The lax limit comes with a lax cone

$$\begin{tikzcd} & \mathcal{V}^2_F \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{V} \arrow[rr, "F"'] & \arrow[u, phantom, "\Rightarrow"] & \mathcal{N} \end{tikzcd}$$

where p_1, p_2 are given on objects as

$$p_1(X, Y, \eta) = X$$

$$p_2(X, Y, \eta) = Y$$

and the natural transformation encodes η. Let $F_1 = p_2 \colon \mathcal{V}^2_F \to \mathcal{N}$ and $F_2 = \mathrm{cof}(F \circ p_1 \to p_2)$. Explicitly, we thus have

$$F_2(X, Y, \eta) = \mathrm{cof}(\eta) \, .$$

The functors $(F_i \colon \mathcal{V}^2_F \to \mathcal{N})_{i=1,2}$ define a pre-schober on the 2-spider:

The functors F_1, F_2 have right adjoints G_1, G_2, given on objects by

$$\begin{aligned} G_1 \colon \mathcal{N} \to \mathcal{V}^2_F, \quad & Y \mapsto (G(Y), Y, \mathrm{counit} \colon FG(Y) \to Y) \\ G_2 \colon \mathcal{N} \to \mathcal{V}^2_F, \quad & Y \mapsto (0, Y, 0) \, . \end{aligned}$$

We have $F_2 \circ G_1 \simeq \mathrm{cof}(FG \xrightarrow{\mathrm{counit}} \mathrm{id}_{\mathcal{N}})$ which is an equivalence since $F \dashv G$ is spherical. We further have $F_1 \circ G_2 \simeq \mathrm{id}_{\mathcal{N}}$. We leave the verification of the remaining conditions to the reader.

The fiber of F_2 is given by the full subcategory $\mathcal{V} \simeq \mathrm{fib}(F_2) = \{(X, F(X), \mathrm{id}_{F(X)})\} \subset \mathcal{V} \overrightarrow{\oplus}_F \mathcal{N}$ and

$$F_1|_{\mathrm{fib}(F_2)} \simeq F \colon \mathcal{V} \longrightarrow \mathcal{N}$$

recovers the spherical functor F.

Remark 2.9. Generalizing Theorem 2.8, given a pre-schober on the n-spider with underlying spherical functor F, its value at the vertex is equivalent to the lax limit of the diagram $[n] \to \mathrm{St}$

$$\mathcal{V} \xrightarrow{F} \mathcal{N} \xrightarrow{\mathrm{id}_{\mathcal{N}}} \dots \xrightarrow{\mathrm{id}_{\mathcal{N}}} \mathcal{N}\,.$$

There is also an explicit recipe for the functors in the pre-schober on the n-spider. See [CHQ23, Prop. 4.7] for a description, based on results of [Chr22]. This recipe is based on the relative Waldhausen $S_\bullet$-construction [Dyc21].

Remark 2.10. Let $\mathcal{F}_n$ be a pre-schober on the n-spider. For any $j' \in \mathbb{Z}/n\mathbb{Z}$, we have

$$\mathcal{F}_{n-1}^{j'} \simeq \mathcal{F}_{n-1}^{j}\,,$$

and we expect that this equivalence is natural in the sense that for different j, j' the arising equivalences

$$\mathbf{Perv}(\mathrm{Sp}_n) \simeq \mathbf{Perv}(\mathrm{Sp}_{n-1})$$

are equivalent.

We propose the following notion of perverse schober $\mathbb{C}$ stratified at 0:

> A perverse schober on $(\mathbb{D}, 0)$ consists of a compatible collection of pre-schobers on all the possible n-spiders linearly embedded in $\mathbb{C}$.

The compatibility means that if we an m-spider embedded in an n-spider embedded in $\mathbb{C}$, then the corresponding pre-schober on the m-spider is obtained from the corresponding pre-schober on the n-spider by passing to the fibers corresponding to the missing edges. The collection of linear embeddings of spiders into $\mathbb{C}$ forms a filtered poset category $\mathrm{Sp}(\mathbb{C})$, and pre-schobers on the n-spider assemble into a functor

$$\mathbf{Perv}(\text{-})\colon\ \mathrm{Sp}(\mathbb{C})^{\mathrm{op}} \to \mathrm{Cat}_{(\infty,2)}\ .$$

Importantly, since it is a filtered category, $\mathrm{Sp}(\mathbb{C})^{\mathrm{op}}$ is weakly contractible.

A merit of having multiple equivalent descriptions of pre-schobers is that they can allow to define sheaf operations, such as a pushforward:

Example 2.11. Consider the $(n:1)$-branched cover $z^n\colon \mathbb{C} \to \mathbb{C}$, branched at the origin. We can embed the n-spider Sp_n in $\mathbb{C}$, such that $z^n(\mathrm{Sp}_n) = \mathrm{Sp}_1 \subset \mathbb{C}$. There is a pushforward functor

$$(z^n)_*\colon \mathbf{Perv}(\mathrm{Sp}_n) \to \mathbf{Perv}(\mathrm{Sp}_1) = \mathbf{Sph}$$

that assigns to each pre-schober on the n-spider $(F_i\colon \mathcal{V}^n \to \mathcal{N}_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ the spherical functor

$$\prod_{i\in\mathbb{Z}/n\mathbb{Z}} F_i\colon \mathcal{V}^n \longrightarrow \prod_{i\in\mathbb{Z}/n\mathbb{Z}} \mathcal{N}_i\,.$$

See [Chr22, Lem. 3.8] for a proof that the above functor is spherical.

2.3 Surfaces and constructible sheaves on graphs

Our next goal is to generalize the notion of pre-schober on an n-spider to pre-schobers on more general graphs embedded in surfaces. After specifying the geometric setup, involving marked surfaces, we discuss constructible sheaves on graphs.

Definition 2.12. A stratified marked surface $(\mathbf{S}, M, P)$ consists of a compact oriented topological surface $\mathbf{S}$ with non-empty boundary $\partial\mathbf{S}$, a finite set of marked points $M \subset \partial\mathbf{S}$, and a collection $P \subset \mathbf{S}^\circ$ of interior points (the 0-dim stratum).

We typically just write $\mathbf{S}$ for the triple $(\mathbf{S}, M, P)$. We sometimes also call P the singularities of $\mathbf{S}$.

We will consider graphs with finite sets of vertices and edges. We do not allow 0-valent vertices. We allow external edges, meaning edges which are incident to only a single vertex and this only once. Each internal edge of a ribbon graph is understood to be composed of two halfedges.

The geometric realization $|\mathbf{G}|$ of a graph $\mathbf{G}$ is the corresponding topological space, obtained by gluing together an interval for every edge along the vertices.

Definition 2.13. We call a graph $\mathbf{G}$ a spanning graph of a stratified marked surface $\mathbf{S}$ if it is equipped with an embedding $i\colon |\mathbf{G}| \subset \mathbf{S}\backslash M$ such that

- i is a homotopy equivalence,
- only the external endpoints of the external edges intersect the boundary $\partial\mathbf{S}$,
- for each boundary component B of $\mathbf{S}$ containing at least one marked point i induces a bijection between the set of external edges of $\mathbf{G}$ intersecting B and the connected components of $B\backslash M$, and
- the 0-dim stratum P is contained in the set of the vertices of $\mathbf{G}$.

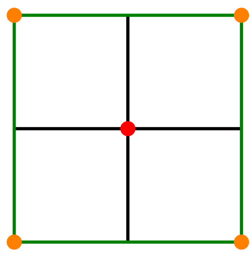
We consider spanning graphs up to isotopy relative $M \cup P$.

Remark 2.14. A ribbon graph is a graph $\mathbf{G}$ equipped with a cyclic order on any set of the half-edges incident to given vertex of $\mathbf{G}$.

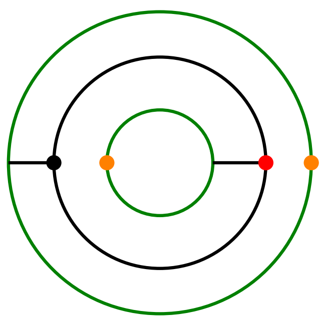
If $\mathbf{G}$ is a spanning graph of a marked surface $\mathbf{S}$, then it inherits a canonical ribbon graph structure via the counterclockwise order induced by the orientation of $\mathbf{S}$.

As noted in Definition 2.2, the n-spider Sp_n defines a ribbon graph.

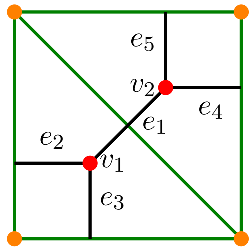
Example 2.15. (1) The 4-spider Sp_4 (in black) as a spanning graph of the 4-gon with a singularity at the center (in red).



(2) The annulus (in green) with two marked points (in orange) and one singularity (in red) together with a spanning graph (in black).



(3) A triangulation of the 4-gon with two singularities and the dual trivalent spanning graph.



Note that every spanning graph is dual to a decomposition of the marked surface into polygons with corners at the marked points.

Definition 2.16. The exit path category $\mathrm{Exit}(\mathbf{G}) \in \mathrm{Cat}_\infty$ of a graph $\mathbf{G}$ is defined as the nerve of the 1-category with

- objects the vertices and edges of $\mathbf{G}$ and
- non-identity morphisms of the form $v \to e$ with v a vertex incident to an edge e. If e is a loop at v, then there are two morphisms $v \to e$ in $\mathrm{Exit}(\mathbf{G})$.

Note that $\mathrm{Exit}(\mathbf{G})$ is a bipartite poset.

Example 2.17. The exit path category of the trivalent spanning graph from Example 2.15.(3) can be depicted as follows:

$$\begin{tikzcd}
e_2 & & & & e_5 \\
& v_1 \arrow[ul] \arrow[dl] \arrow[dr] & & v_2 \arrow[ur] \arrow[dl] \arrow[dr] & \\
e_3 & & e_1 & & e_4
\end{tikzcd}$$

The exit path category of a graph is equivalent to a special case of a more general definition, taking as input any stratified space, see [Lur17, Section A.6]. A sheaf on a stratified space is called constructible if its restriction to each stratum is a locally constant sheaf.

Theorem 2.18 ([Lur17] for $\mathcal{C} = \mathcal{S}$, [PT22])**.** *Let X be a sufficiently nice conically stratified space and $\mathcal{C}$ a compactly generated ∞-category. Then there exists an equivalence of ∞-categories*

$$\mathrm{Shv}_c(X, \mathcal{C}) \simeq \mathrm{Fun}(\mathrm{Exit}(X), \mathcal{C})$$

between the ∞-category of $\mathcal{C}$-valued constructible sheaves on X and the ∞-category of functors from the exit path ∞-category of X to $\mathcal{C}$.

Specializing to $X = \mathbf{G}$ a graph and $\mathcal{C} = \mathrm{St}$ we obtain:

A functor $\mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$ describes a constructible sheaf of stable ∞-categories on $\mathbf{G}$.

Example 2.19. Denote the vertex of the n-spider Sp_n by v and the incident edges by $e_1, \dots, e_n$. A constructible sheaf valued in St on the n-spider Sp_n amounts to a functor $\mathcal{F}_{\mathrm{Sp}_n} \colon \mathrm{Exit}(\mathrm{Sp}_n) \to \mathrm{St}$, amounting to the following data:

- A stable ∞-category $\mathcal{V}^n := \mathcal{F}_{\mathrm{Sp}_n}(v)$.

- n many stable ∞-categories $\mathcal{N}_i \coloneqq \mathcal{F}_{\mathrm{Sp}_n}(e_i)$, $i \in [1, n]$.
- n many functors $F_i \coloneqq \mathcal{F}_{\mathrm{Sp}_n}(v \to e_i)\colon \mathcal{V}^n \to \mathcal{N}_i$, $i \in [1, n]$.

A pre-schober on the n-spider in the sense of Definition 2.3 is thus a constructible sheaf on the n-spider satisfying certain properties.

2.4 Contractions and perverse schobers on surfaces

Definition 2.20 ([Chr22, CHQ23])**.** Let $\mathbf{G}$ be a ribbon graph. A pre-schober parametrized by $\mathbf{G}$ is a functor $\mathcal{F}_{\mathbf{G}}\colon \operatorname{Exit}(\mathbf{G}) \to \mathrm{St}$ such that, for each n-valent vertex v of $\mathbf{G}$, the restriction of $\mathcal{F}_{\mathbf{G}}$ to $\operatorname{Exit}(\mathrm{Sp}_n) \simeq \operatorname{Exit}(\mathbf{G})_{v/}$ defines a pre-schober on Sp_n in the sense of Definition 2.3.

The above definition is summarized as follows:

A pre-schober parametrized by $\mathbf{G}$ is a functor $\operatorname{Exit}(\mathbf{G}) \to \mathrm{St}$ subject to local conditions.

Given a stratified marked surface $\mathbf{S}$ for which $\mathbf{G}$ is a spanning graph, the above definition of $\mathbf{G}$-pre-schober is not accounting for the singularity data of $\mathbf{S}$: the perverse schober needs to be non-singular, also called locally constant, near non-singular vertices. This is defined below as the vanishing of the categorified vanishing cycles. Note that the vanishing cycles of a perverse sheaf on $\mathbb{C}$ vanish if and only if the perverse sheaf is locally constant.

Definition 2.21.

(1) A perverse schober on the 1-spider $F\colon \mathcal{V} \to \mathcal{N}$ is called non-singular if $\mathcal{V} \simeq 0$.

(2) Let $n \neq 2$ and $\mathcal{F}^n \in \mathbf{Perv}(\mathrm{Sp_n})$. We call $\mathcal{F}^n$ non-singular if its image in $\mathbf{Perv}(\mathrm{Sp}_1) = \mathbf{Sph}$ under the equivalence $\mathbf{Perv}(\mathrm{Sp_n}) \simeq \mathbf{Sph}$ from Theorem 2.7 is a non-singular pre-schober on Sp_1.

(3) Let $\mathbf{G}$ be a ribbon graph and P a subset of the set of vertices of $\mathbf{G}$. We denote by $\mathbf{Perv}(\mathbf{G}, P) \subset \mathbf{Perv}(\mathbf{G})$ the full subcategory of perverse schobers which are non-singular away from P.

Example 2.22. The non-singular pre-schober on the 3-spider corresponding to the spherical functor $0\colon 0 \to \mathcal{N}$ is given, up to equivalence, by the diagram

$$\left(F_i\colon \mathcal{N} \oplus_{\mathrm{id}_{\mathcal{N}}}^{\to} \mathcal{N} \simeq \operatorname{Fun}([2], \mathcal{N}) \longrightarrow \mathcal{N}\right)_{i \in \mathbb{Z}/3\mathbb{Z}}$$

with

$$\begin{aligned} F_1 = p_1 &\colon \operatorname{Fun}([2], \mathcal{N}) \to \mathcal{N}, (x_1 \xrightarrow{\eta} x_2) \mapsto x_1 \\ F_2 = \operatorname{cof} &\colon \operatorname{Fun}([2], \mathcal{N}) \to \mathcal{N}, (x_1 \xrightarrow{\eta} x_2) \mapsto \operatorname{cof}(\eta) \\ F_3 = p_2 &\colon \operatorname{Fun}([2], \mathcal{N}) \to \mathcal{N}, (x_1 \xrightarrow{\eta} x_2) \mapsto x_2 \,. \end{aligned}$$

To relate pre-schobers parametrized by different graphs, we consider contractions of edges of graphs. We allow only the contraction of interior edges which are not loops, as such contractions do not change the homotopy type of the graph. Otherwise one is lead to collapses of subsurfaces. Such collapses are also possible, see [FL25].

So ,if $\mathbf{G}$ is a spanning graph of $\mathbf{S}$ and $\mathbf{G}'$ is obtained by a contraction of edges from $\mathbf{G}$, we obtain a canonical embedding of $\mathbf{G}'$ into $\mathbf{S}$, exhibiting $\mathbf{G}'$ as a spanning graph of $\mathbf{S}$, provided that the contraction did not contract any singularities.

Definition 2.23 (Definition sketch)**.** Let $\mathbf{S}$ be a stratified marked surface with 0-dim stratum P. We denote by $\mathrm{Rib}(\mathbf{S})$ the category with

- objects given by spanning graph $\mathbf{G}$ of $\mathbf{S}$ (containing P as vertices),
- and morphisms given by embedded contractions which do not contract any edges connecting two vertices in P.

The category $\mathrm{Rib}(\mathbf{S})$ is in fact a poset category, meaning that there is (up to isotopy) at most one way to contract a spanning graph $\mathbf{G}$ to a different spanning graph $\mathbf{G}'$. A further deep result is the following:

Theorem 2.24. *The category* $\mathrm{Rib}(\mathbf{S})$ *is weakly contractible.*

Pre-schobers on graphs can be pushed along contractions of ribbon graphs:

Theorem 2.25 ([ACJ])**.** *Let* $\mathbf{S}$ *be a stratified marked surface. There exists a functor*

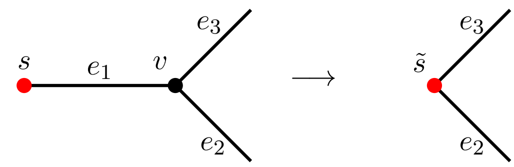
$$\mathbf{Perv}(\text{-}, P)\colon \mathrm{Rib}(\mathbf{S}) \longrightarrow \mathrm{Cat}_{(\infty,2)}$$

assigning to each spanning graph $\mathbf{G}$ *of* $\mathbf{S}$ *the* $(\infty, 2)$*-category of* $\mathbf{G}$*-parametrized pre-schobers which are non-singular away from* P*.*

Further, $\mathbf{Perv}(\text{-}, P)$ *maps contractions to equivalences of* $(\infty, 2)$*-categories.*

The above theorem is inspired by the construction of 2-periodic topological Fukaya categories of Dyckerhoff–Kapranov [DK18], which is a similar construction one categorical level lower.

Example 2.26. We illustrate the action of the equivalence $c_*\colon \mathbf{Perv}(\mathbf{G}) \simeq \mathbf{Perv}(\mathbf{G}')$ arising from a contraction $c\colon \mathbf{G} \to \mathbf{G}'$ in the case of the collapse of an edge connecting a 1-valent singularity with a 3-valent non-singular vertex. We consider the following contraction:

$c\colon$ 

Then

$$\mathrm{Exit}(\mathbf{G}) = \begin{tikzcd} & & & e_3 \\ s \arrow[r] & e_1 & v \arrow[l] \arrow[ur] \arrow[dr] & \\ & & & e_2 \end{tikzcd}$$

and

$$\mathrm{Exit}(\mathbf{G}) = \begin{tikzcd} & e_3 \\ \tilde{s} \arrow[ur] \arrow[dr] & \\ & e_2 \end{tikzcd}$$

A $\mathbf{G}$-parametrized perverse schober $\mathcal{F}_{\mathbf{G}}$ is up to equivalence given by a diagram of the form

$$\begin{tikzcd} & & & \mathcal{N} \\ \mathcal{V} \arrow[r, "F"] & \mathcal{N} & \mathcal{N} \oplus_{\mathrm{id}_{\mathcal{N}}}^{\rightarrow} \mathcal{N} \arrow[l, "p_1"'] \arrow[ur, "p_2"] \arrow[dr, "\mathrm{cof}"] & \\ & & & \mathcal{N} \end{tikzcd}$$

with F a spherical functor. The value of the $\mathbf{G}'$-parametrized perverse schober $c_*(\mathcal{F}_{\mathbf{G}})$ at the contracted vertex is obtained by 'passing to the pullback'

$$\mathcal{V} \times_{\mathcal{N}} \left(\mathcal{N} \oplus_{\mathrm{id}_{\mathcal{N}}}^{\rightarrow} \mathcal{N}\right) \simeq \mathcal{V} \oplus_{F}^{\rightarrow} \mathcal{N}\,.$$

In total, $c_*(\mathcal{F}_{\mathbf{G}})$ thus amounts to the diagram

$$\begin{tikzcd} & \mathcal{N} \\ \mathcal{V} \oplus_{F}^{\rightarrow} \mathcal{N} \arrow[ur, "p_2"] \arrow[dr, "\mathrm{cof}(\eta)"] & \\ & \mathcal{N} \end{tikzcd}$$

Remark 2.27. An explicit formula for pushing a pre-schober along a contraction of ribbon graphs $\mathbf{G} \to \mathbf{G}'$, generalizing Theorem 2.26, can be found in [Chr22, Lem. 4.26, Prop. 4.28]. In short, the push is given by a right Kan extension along an induced functor $\mathrm{Exit}(\mathbf{G}) \to \mathrm{Exit}(\mathbf{G}')$. More explicitly, if $c\colon \mathbf{G} \to \mathbf{G}'$ contracts a single edge e connecting two vertices v_1, v_2 of $\mathbf{G}$ and $\mathcal{F}_{\mathbf{G}}$ is a $\mathbf{G}$-parametrized pre-schober, then we obtain a $\mathbf{G}'$-parametrized pre-schober $c_*(\mathcal{F}_{\mathbf{G}})$ by declaring $c_*(\mathcal{F}_{\mathbf{G}})(v) = \mathcal{F}_{\mathbf{G}}(v_1) \times_{\mathcal{F}_{\mathbf{G}}(e)} \mathcal{F}_{\mathbf{G}}(v_2)$, where v is the vertex of $\mathbf{G}'$ arising from the contraction.

Using Theorem 2.25, we can define perverse schobers on surfaces as follows:

Definition 2.28. Let $\mathbf{S}$ be a marked surface. We define the $(\infty, 2)$-category $\mathbf{Perv}(\mathbf{S})$ of perverse schobers on $\mathbf{S}$ as the colimit of the functor

$$\mathbf{Perv}(\text{-}, P)\colon\ \mathrm{Rib}(\mathbf{S}) \longrightarrow \mathrm{Cat}_{(\infty,2)}\ .$$

Remark 2.29. The fact that $\mathrm{Rib}(\mathbf{S})$ is weakly contractible implies that there exists a canonical equivalence $\mathbf{Perv}(\mathbf{G}, P) \simeq \mathbf{Perv}(\mathbf{S})$ for all spanning graphs $\mathbf{G}$ of $\mathbf{S}$.

> A perverse schober on a marked surface $\mathbf{S}$ consists of a compatible family of pre-schobers parametrized by all possible spanning graphs $\mathbf{G}$ of $\mathbf{S}$.

Further, a perverse schober on a marked surface is determined uniquely up to contractible choice by a single $\mathbf{G}$-parametrized pre-schober for any choice of spanning graph $\mathbf{G}$.

Other approaches to defining perverse schobers on surfaces may be possible, see for instance the Milnor sheaves of [DKS23].

Explicit computations describing a collection of pre-schobers on a surface that are compatible and thus arise from a schober on the surface appear in [Chr21, Section 7.2] and [CC26, Lem. 2.12].

Example 2.30. Let $\beta \in \mathrm{Br}_{n+1}^+$ be a positive braid word (i.e. we impose no braid relations). A Demazure weave $\mathfrak{w}\colon \beta \to \beta'$ is a list of braid 4- and 6-moves as well as the 3-move $s_i^2 \to s_i$. Fix such a Demazure weave $\mathfrak{w}$ and let m be its number of 3-moves. Then [CC26] associates with $\mathfrak{w}$ a pre-schober on a linear spanning graph of the 1-gon $\mathbb{D}$ with m singularities.

Any two weaves $\mathfrak{w}\colon \beta \to \beta'$ can be connected via a sequence of weave equivalences and weave mutations. These induce different embeddings of the linear graph in $\mathbb{D}$, related by Dehn twists. Weaves $\beta \to \beta'$ so cover a subset of all possible linear spanning graphs of $\mathbb{D}$ and give rise to a compatible family of pre-schobers parametrized by these.

In summary, each pair of braid words β, β' leads to a uniquely defined perverse schober on the 1-gon $\mathbb{D}$.

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