Perverse schobers in representation theory

Lecture notes based on a short course at YMSC/BIMSA, Beijing, spring 2022

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Summary

The goal of this short course is to give an introduction to perverse schobers on surfaces and explain applications to the representation theory of dg-algebras such as Ginzburg algebras.

Perverse schobers are a (in general conjectural) categorification of perverse sheaves. We begin in the first lecture by discussion how the relative cohomology of a marked surface with coefficients in a local system, or more generally a perverse sheaf, can be described in terms of constructible sheaves on graphs. In lectures 2-5, will see how this description can be categorified leading to a description of perverse schobers on marked surfaces in terms of constructible sheaves of stable ∞-categories on graphs. In the three remaining lectures, we describe examples of perverse schobers in representation theory and how perverse schobers yield powerful local-to-global methods to study their ∞-categories of global sections. We apply these local-to-global methods to describe so called geometric models for the ∞-categories of global sections of perverse schobers. During the course, we will also introduce and apply a number of concepts from the theory of ∞-categories, such as stable ∞-categories and the ∞-categorical Grothendieck construction.

The target audience will have some experience working with dg-categories, but not necessarily with stable ∞-categories. Some background in the representation theory of quiver algebras will also be helpful in the later lectures.

I sometimes update these notes to include recent progress on perverse schobers.

Acknowledgements

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Contents

Lecture 1: Constructible sheaves on graphs and perverse sheaves

In this lecture, we discuss how the relative cohomology of a marked surface can be computed in terms of the global sections of sheaves on graphs. We also discuss a generalization to relative cohomology with coefficients in a perverse sheaf. Lectures 2-5 will deal with the categorification of this story.

Sheaf cohomology

Let X be a nice (paracompact, locally contractible) topological space, for example a surface or a graph. We fix a field k and consider the category of $\mathrm{Vect}_k^{\mathrm{fin}}$-valued sheaves $\mathrm{Shv}_k(X)$ on X. We denote by $\mathrm{Loc}(X) \subset \mathrm{Shv}_k(X)$ the subcategory of locally constant sheaves, also called local systems.

Let $U \subset X$ be a subset and $W \subset U$. Recall that sheaf cohomology and relative sheaf cohomology define functors, with $i \in \mathbb{N}$,

$$H^i(U;\text{-}), H^i(U,W;\text{-}) : \mathrm{Shv}_k(X) \to \mathrm{Vect}_k^{\mathrm{fin}}\,,$$

satisfying the following properties. Let $F \in \mathrm{Shv}_k(X)$ be a sheaf.

(1) (**Long exact sequence for relative cohomology**) There is a long exact sequence

$$0 \to H^0(U,W;F) \to H^0(U;F) \to H^0(W;F) \to H^1(U,W;F) \to \dots$$

(2) (**Gluing**) Let $U_1, U_2 \subset X$ be closed subsets with $U_1 \cup U_2 = X$. Let $W_1 \subset U_1$ and $W_2 \subset U_2$ also be closed subsets. Then there is the Mayer-Vietoris long exact sequence

$$\begin{aligned} 0 \to H^0(X,W_1\cup W_2;F) &\to H^0(U_1,W_1;F) \oplus H^0(U_2,W_2;F) \to H^0(U_1\cap U_2, W_1\cap W_2;F) \\ &\to H^1(X,W_1\cup W_2;F) \to \dots \end{aligned}$$

(3) (**Homotopy invariance**) Let L be local system, $U' \subset U \subset X$ and $W' \subset W \cap U'$, such that $U' \subset U$ and $W' \subset W$ are homotopy equivalences. Then the map $H^*(U',W';L) \to H^*(U,W;L)$ is an equivalence.

(4) (**Comparison with singular cohomology**) For $N \in \mathrm{Vect}_k^{\mathrm{fin}}$, let $\underline{N}$ be the constant local system on X with value N.

$$H^*(X;\underline{N}) \simeq H^*_{\mathrm{sing}}(X;N) \qquad H^*(X,W;\underline{N}) \simeq H^*_{\mathrm{sing}}(X,W;N)\,.$$

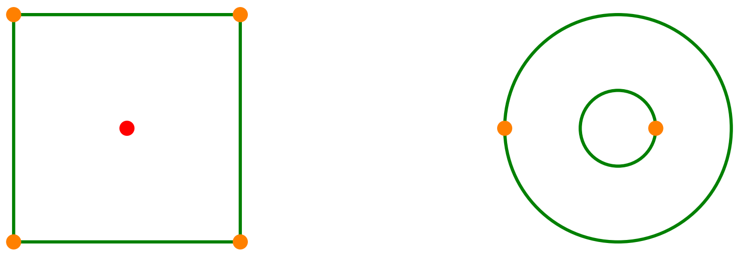
Marked surfaces

If the topological space is a marked surface, then the relative sheaf cohomology of locally constant sheaves admits a particularly simple description.

Definition 1.1. By a surface $\mathbf{S}$, we mean an oriented, compact, connected 2-dimensional manifold with nonempty boundary $\partial\mathbf{S}$. All connected boundary components of $\mathbf{S}$ are circles.

A marked surface $(\mathbf{S}, M)$, mostly simply denoted by $\mathbf{S}$, consists of a surface with a finite nonempty subset $M \subset \mathbf{S}$ of marked points, satisfying that each boundary component contains a marked point. Marked points in the interior $\mathbf{S}^\circ$ of $\mathbf{S}$ are called punctures. The set of punctures is denoted by P.

Example 1.2. Two examples of marked surfaces: the once-punctured 4-gon (disc with 4 boundary marked points and one puncture) and the annulus with two marked points.



All marked surfaces surfaces are obtained from a closed oriented surface of genus g by removing $k \geq 1$ open discs and adding marked points to the k boundary circles and possibly adding punctures.

Given such a marked surface $\mathbf{S}$ without punctures, we are interested in the relative singular cohomology groups $H^*(\mathbf{S}, M; L)$ with coefficients in a local system L.

Definition 1.3. Let $\mathbf{S}$ be a marked surface and $\Gamma \subset \mathbf{S}$ a graph. The Γ is called a spanning graph if

- $\Gamma \subset \mathbf{S}$ is a homotopy equivalence,
- $\Gamma \cap \partial\mathbf{S} \subset \partial\mathbf{S} \backslash (\partial\mathbf{S} \cap M)$ is a homotopy equivalence and
- each puncture is a vertex of Γ.

Exercise 1. Every marked surface admits a spanning graph.

We fix a spanning graph Γ of a marked surface $\mathbf{S}$ without punctures. We observe that $M \subset \mathbf{S}\backslash\Gamma$ is a homotopy equivalence. We thus have $H^*(\mathbf{S}, M; L) \simeq H^*(\mathbf{S}, \mathbf{S}\backslash\Gamma; L)$.

Let $L \in \mathrm{Loc}(\mathbf{S})$ with stalk $N \in \mathrm{Vect}_k^{\mathrm{fin}}$. For all $i \in \mathbb{N}$, we denote by $\underline{H}^i_\Gamma(L)$ the sheaf of k-vector spaces on Γ, whose value on an open set $W \subset \Gamma$ is given by the following colimit over all open $U \subset \mathbf{S}$ with $W \subset U$.

$$\underline{H}^i_\Gamma(L)(W) = \mathrm{colim}_U \, H^i(U, U\backslash\Gamma; L)\,.$$

The sheaves $\underline{H}^*_\Gamma(L)$ compute the (derived) sections of L with support on Γ.

We compute the stalks of $\underline{H}^i_\Gamma(L)$.

Let p be a point on an edge of Γ and U a sufficiently small neighborhood as in Figure 1. Then there is an exact sequence:

$$0 \to \underline{H}^0_\Gamma(L)(U \cap \Gamma) \hookrightarrow \underbrace{H^0(U; L)}_{\simeq N} \to \underbrace{H^0(\underbrace{U\backslash\Gamma}_{\simeq * \amalg *}; L)}_{\simeq N^{\oplus 2}} \to \underline{H}^1_\Gamma(L)(U) \to 0 \to \dots$$

The stalk at p is thus given by $\underline{H}^1_\Gamma(L)_p \simeq N$ and $\underline{H}^i_\Gamma(L)_p \simeq 0$ for $i \neq 1$.

Let v be a vertex of Γ of valency n. Again applying the long exact sequence, one shows that the stalk at v is given by $\underline{H}^1_\Gamma(L)_v \simeq N^{\oplus n-1}$ and $\underline{H}^i_\Gamma(L)_v \simeq 0$ for $i \neq 1$.

Constructible sheaves on graphs

We observe that the sheaf $\underline{H}^i_\Gamma(L)$ is locally constant on each edge of Γ, we call such a sheaf a constructible sheaf on Γ. It can be fully encoded by

- for each edge e of Γ its stalk $\underline{H}^i_\Gamma(L)_e$ at any point on e.
- the stalks $\underline{H}^i_\Gamma(L)_v$ at the vertices v of Γ.

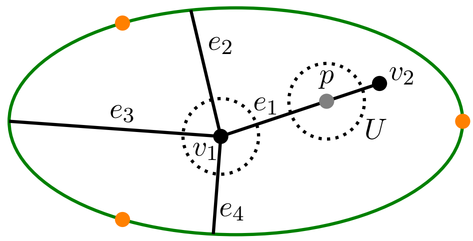
- the restriction maps $\underline{H}^i_\Gamma(L)_v \to \underline{H}^i_\Gamma(L)_e$ arising from incidence of e and v.

Definition 1.4. Let Γ be a graph. We denote by $\mathrm{Exit}(\Gamma)$ the category determined as follows.

- The objects of $\mathrm{Exit}(\Gamma)$ are the vertices and edges of Γ.
- There is a morphism $v \to e$, whenever v is a vertex and e an edge of Γ incident to v. If e both begins and ends at v, then there are two distinct morphisms $v \to e$.

We call $\mathrm{Exit}(\Gamma)$ the exit path category.

Definition 1.5. Let Γ be a graph. A constructible sheaf on Γ with values in a category $\mathcal{C}$ is a functor $\mathrm{Exit}(\Gamma) \to \mathcal{C}$.



$$\mathrm{Exit}(\Gamma) = \begin{tikzcd} & e_2 & & v_2 \arrow[dl] \\ e_3 & v_1 \arrow[l] \arrow[u] \arrow[r] \arrow[d] & e_1 & \\ & e_4 & & \end{tikzcd}$$

Figure 1: On the left: a spanning graph Γ for the 3-gon with four edges, three of which are external, and two vertices. The point $p \in \Gamma$ lies on the edge e_1 and U is a small neighborhood. On the right: the exit path category of Γ.

Definition 1.6. Let $F : \Gamma \to \mathcal{C}$ be a constructible sheaf with values in $\mathcal{C}$. The global sections $H(\Gamma; F) \in \mathcal{C}$ of F are defined as the limit of F.

Note that $H(\Gamma, \underline{H}^i_\Gamma(L))$ is equivalent to the (usual, not derived) global sections of $\underline{H}^i_\Gamma(L)$ as a sheaf on Γ.

Example 1.7. In the example from Figure 1, the constructible sheaf $\underline{H}^1_\Gamma(L)$ amounts to a diagram of the form

$$\begin{tikzcd} & N & & 0 \arrow[dl] \\ N & N^{\oplus 3} \arrow[l] \arrow[u] \arrow[r] \arrow[d] & N & \\ & N & & \end{tikzcd}$$

in $\mathrm{Vect}_k^{\mathrm{fin}}$. Its limits is equivalent to $N^{\oplus 2}$, which is equivalent to the first relative singular cohomology of the 3-gon. All other relative singular cohomology groups vanish.

Theorem 1.8. *For all i, there exist isomorphisms*

$$H(\Gamma; \underline{H}^i_\Gamma(L)) \simeq H^i(\mathbf{S}, \mathbf{S}\backslash\Gamma; L) \simeq H^i(\mathbf{S}, M; L)\,. \tag{1}$$

The theorem expresses, that the relative cohomology with coefficients in L arise from gluing the local sections of L with support on Γ, as encoded by $\underline{H}^*_\Gamma(L)$.

Remark 1.9. By Lefschetz duality, there exist isomorphisms between singular homology and cohomology $H^i_{\mathrm{sing}}(\mathbf{S}, M) \simeq H^{\mathrm{sing}}_{n-i}(\mathbf{S}, \partial\mathbf{S}\backslash M)$. Using that all boundary components have at least one marked point, one can find an isomorphism $H_i^{\mathrm{sing}}(\mathbf{S}, \partial\mathbf{S}\backslash M) \simeq H_i^{\mathrm{sing}}(\mathbf{S}, M)$. There also exists a homology version of Theorem 1.8 computing $H_i^{\mathrm{sing}}(\mathbf{S}, M)$, where $H(\Gamma, \underline{H}^i_\Gamma(\underline{k}))$ is replaced by the global sections of a constructible cosheaf on Γ (i.e. the colimit of a functor $\mathrm{Exit}(\Gamma)^{\mathrm{op}} \to \mathrm{Vect}_k^{\mathrm{fin}}$) which locally describes local relative homology groups.

Proof sketch of Theorem 1.8. We start with the case of the n-gon $\mathbf{D}$ with $|M| = n$ boundary marked points and the graph Γ consisting of a single vertex with n incident edges. The local system L is up to isomorphism constant, we denote its stalk by N. By the long exact sequence for relative cohomology, we have $H^*(\mathbf{D}, M; \underline{N}) \simeq H^1(\mathbf{D}, M; \underline{N}) \simeq N^{\oplus n-1}$. The constructible sheaf $\underline{H}^i_\Gamma(L)$ vanishes for $i \neq 1$ and is for $i = 1$ given (up to isomorphism) by the following diagram.

$$
\begin{tikzcd}
 & N & \\
\vdots & N^{\oplus n-1} \arrow[l] \arrow[u] \arrow[r] \arrow[d] & N \\
 & N &
\end{tikzcd}
$$

The object $v \in \mathrm{Exit}(\Gamma)$ corresponding to the single vertex of Γ is an initial object and the limit of $\underline{H}^1_\Gamma(L)$ is thus given $\underline{H}^i_\Gamma(L)(v) \simeq N^{\oplus n-1} \simeq H^1(\mathbf{D}, M)$.

The general case is now dealt with by gluing, i.e. by using Mayer-Vietoris. Let $\tilde{\mathbf{S}}$ be an oriented marked surface with a spanning graph $\tilde{\Gamma}$. Suppose that $\tilde{\mathbf{S}}$ is the union of two marked surfaces $\mathbf{S}$ and $\mathbf{S}'$ with spanning graphs $\Gamma = \tilde{\Gamma} \cap \mathbf{S}$ and $\Gamma' = \tilde{\Gamma} \cap \mathbf{S}'$, such that the intersection $\mathbf{S} \cap \mathbf{S}'$ consists of m contractible components $I_1, \ldots, I_m$. Suppose also that the theorem has already been shown for these two surfaces. We denote $e_i = \tilde{\Gamma} \cap I_i$ and consider it as a graph by putting a vertex in the center. We have two exact sequences, which fit into a commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & H(\widetilde{\Gamma}; \underline{H}^1_{\widetilde{\Gamma}}(L)) \arrow[r] & H(\Gamma; \underline{H}^1_\Gamma(L|_{\mathbf{S}})) \oplus H(\Gamma'; \underline{H}^1_{\Gamma'}(L|_{\mathbf{S}'})) \arrow[r] \arrow[d, "\simeq"] & \bigoplus_{i=1}^m H^1(e_i; \underline{N}) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & H^1(\tilde{\mathbf{S}}, \tilde{\mathbf{S}}\backslash\tilde{\Gamma}; L) \arrow[r] & H^1(\mathbf{S}, \mathbf{S}\backslash\Gamma; L) \oplus H^1(\mathbf{S}', \mathbf{S}'\backslash\Gamma'; L) \arrow[r] & \bigoplus_{i=1}^m H^1(I_i, I_i\backslash\tilde{\Gamma}; L) \arrow[r] & 0
\end{tikzcd}
$$

From this, we can find an isomorphism $H(\widetilde{\Gamma}; \underline{H}^1_{\widetilde{\Gamma}}(L)) \simeq H^1(\widetilde{\mathbf{S}}, \widetilde{\mathbf{S}}\backslash\widetilde{\Gamma}; L)$. This argument also shows that for $i \neq 1$, all vector spaces in (1) vanish and are thus isomorphic. □

Perverse sheaves on surfaces

Let $(\mathbf{S}, M)$ be a marked surface, possibly with punctures. The category $\mathrm{Shv}_k(\mathbf{S})$ is abelian and we form its bounded derived category $\mathcal{D}^b(\mathrm{Shv}_k(\mathbf{S}))$. Given $F \in \mathcal{D}^b(\mathrm{Shv}_k(\mathbf{S}))$, we denote by $H^i(F) \in \mathrm{Shv}_k(\mathbf{S})$ the i-th cohomology sheaf.

Definition 1.10. Denote by $\mathrm{Shv}_k(\mathbf{S})$ the abelian category of $\mathrm{Vect}_k^{\mathrm{fin}}$-valued sheaves on $\mathbf{S}$. Let $F \in \mathcal{D}^b(\mathrm{Shv}_k(\mathbf{S}))$. F is called a perverse sheaf with singularities at most at P if it satisfies the following.

- Let $i\colon \mathbf{S}\backslash P \hookrightarrow \mathbf{S}$. We have $H^i(i^*F) = H^i(F|_{\mathbf{S}\backslash P}) = 0$ for $i \neq 0$ and $H^0(F)|_{\mathbf{S}\backslash P}$ is a local system. Note that this implies $H^i(i^!F) \simeq 0$ for $i \neq 2$.
- Let $p \in P$ be a puncture and $j_p\colon \{p\} \hookrightarrow \mathbf{S}$ the inclusion. Then $H^i(j_p^!(F)) = 0$ for $i > 2$ and $H^i(j_p^*(F)) = H^i(F|_p) = 0$ for $i < 0$.

One can show that $H^i(F) = 0$ for all $i \neq 0, 1$.

Perverse sheaves on $\mathbf{S}$ form an abelian category denoted $\mathrm{Perv}(\mathbf{S})$. There exists a fully faithful functor $\mathrm{Loc}(\mathbf{S}) \hookrightarrow \mathrm{Perv}(\mathbf{S})$. As an introduction to perverse sheaves, which conveys much intuition, we recommend [Wil15].

Remark 1.11. The definition of a perverse sheaf on a marked surface $\mathbf{S}$ does not depend on the number of boundary marked points, but only on the surface itself and the set P of punctures. The boundary marked points are only needed for the algebraic descriptions of the category $\mathrm{Perv}(\mathbf{S})$, see the following Theorems and Theorem 1.14 below.

Let $j\colon |\Gamma| \hookrightarrow \mathbf{S}$ be the inclusion of the spanning graph Γ. Given a perverse sheaf F on $\mathbf{S}$, the complex of sheaves $j^!(F) \in \mathcal{D}^b(\mathrm{Shv}_k(\Gamma))$ describes derived sections of F with support on Γ. Remarkably, this complex is again pure, meaning that $H^i(j^!F) = 0$ for $i \neq 1$ and $H^1(j^!F)$ is a constructible sheaf of vector spaces on Γ. For a proof, see [KS16, Prop. 3.2]. This generalizes our previous construction for a local system L, since $H^1(j^!(F)) \simeq \underline{H}^1_\Gamma(L)$. We encode $H^1(j^!F)$ as a functor, denoted $\underline{H}^1_\Gamma(F)\colon \mathrm{Exit}(\Gamma) \to \mathrm{Vect}_k^{\mathrm{fin}}$. We call the global sections of $\underline{H}^1_\Gamma(F)$ the cohomology of $\mathbf{S}$ with support on Γ with coefficients in the perverse sheaf F.

The following theorems explain how much information about F is encoded by the constructible sheaf $\underline{H}^1_\Gamma(F)$.

Theorem 1.12 ([GGM85])**.** *Let $\mathbf{D}$ be the once-punctured 1-gon. Let $\mathcal{A}_1$ be the abelian category of diagrams of finite dimensional vector spaces*

$$V^1 \underset{s}{\overset{r}{\rightleftarrows}} N$$

satisfying that $sr - \mathrm{id}_N$ and $rs - \mathrm{id}_{V^1}$ are invertible.

Let Γ_1 be the graph depicted on the left in Figure 2, also called the 1-spider. There is an equivalence of categories:

$$\begin{aligned} \mathrm{Perv}(\mathbf{D}) &\longrightarrow \mathcal{A}_1 \\ F &\longmapsto \underline{H}^1_{\Gamma_1}(F)(v) \underset{\delta}{\overset{\mathrm{res}}{\rightleftarrows}} \underline{H}^1_{\Gamma_1}(F)(e) \end{aligned}$$

Here $\mathrm{res} = \underline{H}^1_\Gamma(F)(v \to e)$ is the restriction map. Let $i\colon \{x\} \hookrightarrow \mathbf{D}$ be the inclusion of a point $x \in \partial D \backslash \Gamma$. Then

$$\delta\colon \underline{H}^1_\Gamma(F)(e) \simeq H^1(i_*i^*F)(\mathbf{D}) \to H^0(j_!j^!F)(\mathbf{D}) \simeq \underline{H}^1_\Gamma(F)(v)$$

is the connecting homomorphism arising from the distinguished triangle $j_!j^!(F) \to F \to i_*i^*(F)$.

V^1 is called the vector space of vanishing cycles and N is called the vector space of nearby cycles.

Theorem 1.13 ([KS16])**.** *Let $\mathbf{D}$ be the once-punctured n-gon with $n \geq 2$. Let $\mathcal{A}_n$ be the abelian category whose objects correspond to diagrams of finite dimensional vector spaces*

$$\left(V^n \underset{s_i}{\overset{r_i}{\rightleftarrows}} N_i \right)_{1 \leq i \leq n}$$

satisfying that $r_i \circ s_i = \mathrm{id}_{N_i}$, $r_i \circ s_{i+1}$ is invertible (with i modulo n) and $r_i \circ s_j = 0$ else.

Let Γ_n be the graph on the right in Figure 2, also called the n-spider. There is an equivalence of categories:

$$\begin{aligned} \mathrm{Perv}(\mathbf{D}) &\longrightarrow \mathcal{A}_n \\ F &\longmapsto \left(\underline{H}^1_{\Gamma_n}(F)(v) \underset{\delta}{\overset{\mathrm{res}}{\rightleftarrows}} \underline{H}^1_{\Gamma_n}(F)(e_i) \right)_{1 \leq i \leq n} \end{aligned}$$

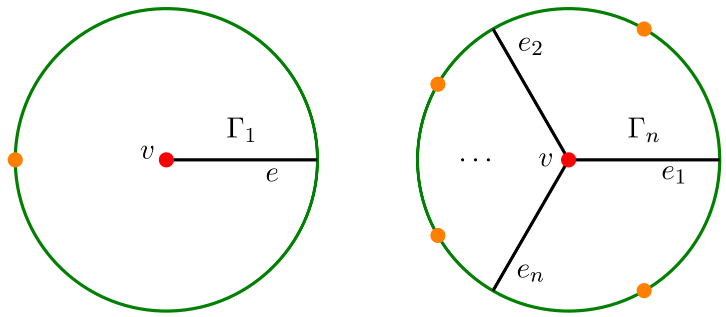


Figure 2: On the left: the once-punctured 1-gon with a spanning graph. On the right: the once-punctured n-gon with a spanning graph with $n \geq 2$.

Using that being perverse is a local condition, Theorems 1.12 and 1.13 combine to the following:

Theorem 1.14 ([KS16, Theorem 3.6])**.** *Let* $\mathbf{S}$ *be a marked surface with spanning graph* Γ. *Then* $\mathrm{Perv}(\mathbf{S})$ *is equivalent to the abelian category of diagrams*

$$\mathrm{Exit}(\Gamma) \amalg_{\mathrm{ob}(\mathrm{Exit}(\Gamma))} \mathrm{Exit}(\Gamma)^{\mathrm{op}} \to \mathrm{Vect}_k^{\mathrm{fin}}$$

which restrict at each vertex of Γ *and its incident halfedges to a diagram of the form described in Theorems 1.12 and 1.13.*

Remark 1.15. Given a perverse sheaf F, the functor $X\colon \mathrm{Exit}(\Gamma) \amalg_{\mathrm{ob}(\mathrm{Exit}(\Gamma))} \mathrm{Exit}(\Gamma)^{\mathrm{op}} \to \mathrm{Vect}_k^{\mathrm{fin}}$ can be described as follows: the restriction $X|_{\mathrm{Exit}(\Gamma)}$ is given by the constructible sheaf $\underline{H}^1_\Gamma(F)$ of sections of F with support on Γ. The other restriction $X|_{\mathrm{Exit}(\Gamma)^{\mathrm{op}}}$ is given by the constructible cosheaf $\underline{H}^{-1}_\Gamma(\mathbb{D}(F))^*$, where $\mathbb{D}(F)$ denotes the Verdier dual of F and $(\text{-})^*$ denotes the passage to dual vector spaces, which turns constructible sheaves on Γ into constructible cosheaves on Γ. A perverse sheaf on $\mathbf{S}$ may thus be encoded in terms of a constructible sheaf on Γ and a constructible cosheaf on Γ, whose (co)stalks are pointwise identified.

Exercise 2. Let F be a perverse sheaf on the once-punctured n-gon. Show that there exists an isomorphism of vector spaces $\underline{H}^1_{\Gamma_n}(F)(v) \simeq \underline{H}^1_{\Gamma_1}(F)(v) \oplus \underline{H}^1_{\Gamma_1}(F)(e)^{\oplus n-1}$.
Hint: Use the distinguished triangle $j_*j^!(F) \to F \to i_*i^*(F)$ with $j : \Gamma_n \hookrightarrow \mathbf{D}$ and $i : \mathbf{D}\backslash\Gamma_n \hookrightarrow \mathbf{D}$.

Lecture 2: Crash course on stable ∞-categories

In lectures 3 and 4, we will discuss how the description of perverse sheaves on marked surfaces can be categorified, leading to perverse schobers on surfaces. Before that, we need discuss basics from Lurie's theory of stable ∞-categories, which is the topic of this lecture.

∞-categories

Notation 2.1. We denote by Δ the simplex category. A simplicial set is a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$. We write Set_Δ for the category of simplicial sets.

We denote the simplicial set corepresented by $[n] \in \Delta$ by Δ^n.

Given a simplicial set $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$, we denote by $X_0 := X([0])$ its set of 0-simplicies, also called objects (and sometimes also called vertices). We also write $x \in X$ if $x \in X_0$ is an object. We denote by $X_1 := X([1])$ its set of 1-simplicies, also called morphisms (and sometimes also called edges).

We assume that the reader is familiar with the language of simplicial sets and with the following definition.

Definition 2.2. An ∞-category is a simplicial set $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$, which has the right lifting property with respect to the inner horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, $1 \leq i \leq n-1$.

Example 2.3. Given a 1-category $\mathcal{C}$, the nerve $N(\mathcal{C}) \in \mathrm{Set}_\Delta$ is an ∞-category. It is the simplicial set whose n-simplicies are collections of n composable morphisms $a_0 \to a_2 \to \cdots \to a_n$ in $\mathcal{C}$.

Sometimes, wo do not distinguish a category from its nerve.

Example 2.4. Topological spaces (usually simply called space) can be considered as ∞-categories. Given a space X, the associated ∞-category is its singular set $\mathrm{Sing}(X) \in \mathrm{Set}_\Delta$, also called singular complex. It is the simplicial set whose n-simplicies are continuous maps $|\Delta^n| \to X$, where $|\Delta^n| = \{(x_0,\dots,x_n) \in [0,1]^{n+1} | \sum_{i=0}^n x_i = 1\}$. The ∞-category $\mathrm{Sing}(X)$ is further a Kan complex. Kan complexes can be used to model spaces up to homotopy equivalences.

We write Cat_∞ for the ∞-category of (small) ∞-categories and $\mathcal{S} \subset \mathrm{Cat}_\infty$ for the full subcategory of Kan complexes. The ∞-category $\mathcal{S}$ is called the ∞-category of (small) spaces.

The above examples allow us to unify category theory and homotopy theory into a larger theory of ∞-categories. Most notions from category theory admit a well behaved generalization to the world of ∞-categories. Examples include limits and colimits, Kan extensions, adjunctions,... In the following lectures, the plan is to introduce all needed ∞-categorical background material, which goes beyond the most basic aspects of ∞-category theory. Introductory treatments of ∞-category theory include [Lur23, Cis19]. A comprehensive treatment is given by Lurie in [Lur09, Lur17, Lur18].

Stable ∞-categories

We proceed by introducing stable ∞-categories, which form an enhancement of triangulated categories with excellent formal properties.

Given an ∞-category $\mathcal{C}$ and two objects $x, y \in \mathcal{C}$, we denote by $\mathrm{Map}_{\mathcal{C}}(x,y) \in \mathcal{S}$ the mapping space. Its objects are morphisms $x \to y$ in $\mathcal{C}$. An object $x \in \mathcal{C}$ is called a zero object if $\mathrm{Map}_{\mathcal{C}}(x,y)$ and $\mathrm{Map}_{\mathcal{C}}(y,x)$ are contractible spaces for all $y \in \mathcal{C}$. We write $x = 0$ if x is a zero object.

Definition 2.5. Let $\mathcal{C}$ be an ∞-category with a zero object. $\mathcal{C}$ is called[1] a stable ∞-category if it admits all finite limits and colimits and a commutative square in $\mathcal{C}$ is pullback if and only if it is pushout.

Notation 2.6. We denote a commutative square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ which is both pullback and pushout by a box $\square$ in the center:

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & b \arrow[d] \\ c \arrow[r] & d \end{tikzcd}$$

Such squares are called *biCartesian* squares.

Definition 2.7. We call a commutative square in an ∞-category

$$\begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & b \arrow[d, "\beta"] \\ 0 \arrow[r] & c \end{tikzcd}$$

[1] This definition is by [Lur17, 1.1.3.4] equivalent to the definition of stable ∞-category in [Lur17, 1.1.1.9].

- a fiber sequence if it is pullback. In this case, we write $\mathrm{fib}(\beta) = a$ and call it the fiber of β.
- a cofiber sequence if it is pushout. In this case, we write $\mathrm{cof}(\alpha) = c$ and call it the cofiber of α.

In a stable ∞-category, every morphisms admits a fiber and a cofiber, which is well defined up to equivalence.

If $\mathcal{C}$ is stable, then fiber sequences coincide with cofiber sequences. The homotopy 1-category of a stable ∞-category $\mathcal{C}$ canonically inherits a triangulated structure such that the fiber and cofiber sequences define the distinguished triangles in this triangulated category, see [Lur17, 1.1.2.14]. The cofiber of a morphism is the stable ∞-categorical version of the cone of the morphism.

Remark 2.8. Let $\mathcal{C}$ be an ∞-category with a zero object and which admits all finite limits and colimits. Taking the fiber or cofiber forms exact functors

$$\mathrm{fib}, \mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}\,.$$

These functors can be constructed using the technology of Kan extensions, see [Lur09, 1.1.1.7].

The cone in a triangulated category is infamous for not being functorial. We can thus use stable ∞-categories as an enhancement of triangulated categories, allowing for functorial cones (and also much more). In practice, all triangulated categories of interest arise as the homotopy categories of stable ∞-categories.

We next describe the analog of the triangulated shift.

Definition 2.9. Assume that $\mathcal{C}$ has a zero object and admits all finite limits and colimits.

Let $a \in \mathcal{C}$. Consider the (essentially unique) morphism $a \to 0$. We define the *suspension* (or sometimes called shift) of a as $a[1] \coloneqq \mathrm{cof}(a \to 0)$. Similarly, we define the *delooping* of a as $a[-1] \coloneqq \mathrm{fib}(0 \to a)$.

Suspension and delooping form functors $[1], [-1] : \mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is stable, then these functor are mutually inverse equivalences.

We illustrate the fact that the suspension/delooping functors are mututally inverse. Consider a biCartesian square

$$\begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & b \end{tikzcd} \tag{2}$$

in a stable ∞-category $\mathcal{C}$. The fact that it is pushout means that $b \simeq a[1]$. The fact that it is pullback means that $a \simeq b[-1]$. The fact that any square is pullback if and only if it is pushout thus implies that $a[1][-1] \simeq a$.

Remark 2.10. A commutative square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ describes more than its morphisms and the fact that it commutes. There are specific 2-simplicies exhibiting the commutativity. The importance of this becomes clear in the diagram (2). Any square of the form (2) trivially commutes, because there is only one morphism (up to equivalence) of the form $a \to 0 \to b$. The additional data of the 2-simplicies encodes a self-homotopy of the zero morphism $a \to b$, which encodes the non-trivial datum of a morphism $a[1] \to b$, which is an equivalence.

There exists an equivalence of functors $\mathrm{cof} \simeq \mathrm{fib} \circ [1]$. This can be illustrated on the level of objects as follows. Let $\alpha : a \to b$ be a morphism in a stable ∞-category. We have the following diagram consisting of two biCartesian squares.

$$\begin{tikzcd}
\mathrm{fib}(\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & a \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
0 \arrow[r] & b \arrow[r] & \mathrm{cof}(\alpha)
\end{tikzcd}$$

The pasting laws for pullbacks imply that the outer square is also pullback and thus biCartesian. This shows that $\mathrm{fib}(\alpha)[1] \simeq \mathrm{cof}(\alpha)$. The fiber of a morphism is thus a shifted version of the cone.

We summarize the enhancement of triangulated categories in terms of stable ∞-categories via the following dictionary.

triangulated categories	stable ∞-categories
object	0-simplex/object/vertex
morphism	1-simplex/morphism/edge
shift functor	suspension functor
inverse shift functor	loop functor
distinguished triangle	fiber and cofiber sequence
mapping cone	cofiber
mapping cone shifted by $[-1]$	fiber

Table 1: A small dictionary for translating between triangulated categories and stable ∞-categories.

Examples of stable ∞-categories

- **Derived ∞-categories of abelian categories.** Let $\mathcal{A}$ be a Grothendieck abelian category, in particular admitting enough projective objects. A typical example is the abelian category of modules over a ring. Then there exists a stable ∞-category $\mathcal{D}(\mathcal{A})$ called the unbounded derived ∞-category of $\mathcal{A}$. It can be defined as the ∞-categorical localization of the nerve of the category of chain complexes $\mathrm{Ch}(\mathcal{A})$ at the quasi-isomorphisms.

- **Nerves of dg-categories.** Let $\mathcal{C}$ be a k-linear dg-category (k is the ground field). There are exists an explicitly defined ∞-category $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$, called the dg-nerve of $\mathcal{C}$, see [Lur17, Section 1.3.1]. If $\mathcal{C}$ is pretriangulated, then $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ is stable, see [Fao17].

  Via the dg-nerve, pretriangulated dg-categories can thus be considered as stable ∞-categories. There is a precise comparison result between k-linear, idempotent complete, stable ∞-categories and dg-categories up to Morita equivalence, see [Coh13]. In this way, stable ∞-categories can be seen as more general than dg-categories.

- **Derived ∞-categories of dg-categories.** Let A be a k-linear dg-category. We denote by dgMod_A the dg-category of right dg A-modules, meaning dg-functors $A^{\mathrm{op}} \to \mathrm{Ch}(k)$. The dg-nerve of dgMod_A is (in general) **not** the correct unbounded derived ∞-category of A. Instead, the stable ∞-category $\mathcal{D}(A)$ is defined as the dg-nerve of the full dg subcategory $\mathrm{dgMod}_A^{\mathrm{cf}} \subset \mathrm{dgMod}_A$ consisting of fibrant and cofibrant dg A-modules with respect to the projective model structure (defined e.g. in [Toë07]). A detailed discussion of the case that A is a dg-algebra, i.e. has a single object, was given by the author in [Chr22a, Sections 2.4 and 2.5].

- **Spectra.** The stable ∞-category of spectra is another important example of a stable ∞-category, and in some sense the prime example. We refer to [Lur17, Section 1.4] for more details.

- **Local systems.** Given a space X, the functor category $\mathrm{Fun}(X, \mathcal{D}(k))$ is a stable ∞-category. Note that the ∞-category of functors into a stable ∞-category is again stable by [Lur17, 1.1.3.1]. The functor category $\mathrm{Fun}(X, \mathcal{D}(k))$ contains the nerve of the abelian category of local systems $\mathrm{Loc}(X)$ on X as the full subcategory of functors with values in $\mathrm{Vect}_k^{\mathrm{fin}} \subset \mathcal{D}(k)$. We thus call $\mathrm{Fun}(X, \mathcal{D}(k))$ the ∞-category of local systems.

  The morphism of simplicial sets $\pi : X \to *$ induces a pullback functor

  $$\pi^* : \mathcal{D}(k) \simeq \mathrm{Fun}(*, \mathcal{D}(k)) \longrightarrow \mathrm{Fun}(X, \mathcal{D}(k))$$

  which admits by [Lur09, 4.3.3.7] left and right adjoints $\pi_!$, respectively, π_*. The functor $\pi_!$ maps a functor $F : X \to \mathcal{D}(k)$ to its colimit and π^* maps F to its limit. The functor π_* can be seen as an ∞-categorical version of the cohomology with coefficients in a local system functor and $\pi_!$ can be seen as an ∞-categorical version of the homology with coefficients in a local system functor. This folklore result goes back at least to a lecture by Lurie[2] and also appears in [BD19, Section 5.1.1].

Exact functors

Definition 2.11. A functor between ∞-categories is called exact, if it preserves all finite limits and colimits. For functors between stable ∞-categories, this equivalent to preserving all fiber and cofiber sequences.

Given two ∞-categories $\mathcal{C}, \mathcal{D}$, we denote by $\mathrm{Ex}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by exact functors.

Exact functors between stable ∞-categories induce triangulated functors on the triangulated homotopy categories. In practice, it is convenient to restrict to functors between presentable, stable ∞-categories which preserve all colimits, because of their even more excellent formal properties.

Definition 2.12. An ∞-category $\mathcal{C}$ is called presentable if it admits all colimits and is accessible, meaning that there exists a regular cardinal κ and a small ∞-category $\mathcal{D}$, such that $\mathcal{C} \simeq \mathrm{Ind}_\kappa(\mathcal{D})$. The category Ind_κ is the κ-inductive completion[3] (short: κ-Ind-completion).

When working with presentable ∞-categories, one mostly encounters the case that $\kappa = \omega$ is the countable cardinal in above definition, when $\mathrm{Ind}_\omega = \mathrm{Ind}$ is just the usual Ind-completion. For example, the unbounded derived ∞-category $\mathcal{D}(A)$ of any dg-category A is presentable, because $\mathcal{D}(A) \simeq \mathrm{Ind}(\mathcal{D}^{\mathrm{perf}}(A))$.

Remark 2.13. If $F : \mathcal{A} \to \mathcal{B}$ is an exact functor between stable ∞-categories, then $\mathrm{Ind}(F) : \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$ is a colimit preserving functor between presentable and stable ∞-category. By passing to Ind-completions, we can thus always restrict to Ind-complete stable ∞-categories and colimit preserving functors.

[2] See `https://www.math.ias.edu/~lurie/287xnotes/Lecture26.pdf`.

[3] The κ-inductive completion consists of freely adding κ-filtered colimits, see [Lur09, Section 5.3].

Presentable ∞-categories have many desirable properties. For example, they admit all limits and colimits. The most important one is the following ∞-categorical adjoint functor theorem.

Theorem 2.14 (∞-categorical adjoint functor theorem, [Lur09, 5.5.2.9])**.** *A functor between presentable ∞-categories admits*

- *a right adjoint if and only it preserves all colimits and*
- *a left adjoint if and only it preserves all limits and is accessible (meaning that it preserves κ-filtered colimits for some regular cardinal κ).*

Definition 2.15 ([Lur09, 5.5.0.1.])**.** Let $\mathcal{C}, \mathcal{D}$ be two presentable ∞-categories.

- We denote by $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by functor which preserve all colimits.
- We denote by $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by functor which preserve all limits and filtered colimits.

Exercise 3. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between stable ∞-categories which preserves finite limits or finite colimits. Show that F is exact. Conclude that given two presentable, stable ∞-categories $\mathcal{C}, \mathcal{D}$, there are inclusions $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}), \mathrm{Fun}^R(\mathcal{C}, \mathcal{D}) \subset \mathrm{Ex}(\mathcal{C}, \mathcal{D})$.

Example 2.16. Let $f : A \to B$ be a morphism between dg-algebras. This endows B with the structure of a dg A-module. The tensor dg-functor

$$\text{-} \otimes_A B : \mathrm{dgMod}_A \longrightarrow \mathrm{dgMod}_B \tag{3}$$

can be shown to preserve cofibrant objects and thus restricts to a functor

$$\text{-} \otimes_A B : \mathrm{dgMod}_A^{\mathrm{cf}} \longrightarrow \mathrm{dgMod}_B^{\mathrm{cf}} \,.$$

Passing to dg-nerves, we obtain a colimit preserving functor between stable, presentable ∞-categories, denoted

$$f_! : \mathcal{D}(A) \longleftrightarrow \mathcal{D}(B)\,. \tag{4}$$

One can also write $f_! = \text{-}\otimes_A^L B$. This dg-functor admits a right adjoint $f^* = \mathrm{RHom}_B(B, \text{-})$, which is induced from the dg-functor $\mathrm{Hom}_B(B, \text{-}) : \mathrm{dgMod}_B \to \mathrm{dgMod}_A$. In general, $\mathrm{Hom}_B(B, \text{-})$ does not preserve cofibrant objects (e.g. if $B \in \mathrm{dgMod}_A$ is not cofibrant). In this case, the induced functor f^* is not simply obtained from passing to dg-nerve, but is a derived functor, which is object-wise obtained by taking the cofibrant replacements of the image under $\mathrm{Hom}_B(B, \text{-})$.

The adjointness of $f_!$ and f^* follows from the fact that $\text{-} \otimes_A B \dashv \mathrm{Hom}_B(B, \text{-})$ is a Quillen adjunction and the fact it thus induces an adjunction on the level of the underlying ∞-categories, see [MG16].

The above story can be generalized to dg-categories and dg-bimodules. Let A and B be k-linear dg-categories. A dg A-B-bimodule is a right dg $A^{\mathrm{op}} \otimes_k B$-module. Let $M \in \mathrm{dgMod}_{A^{\mathrm{op}} \otimes_k B}$ be cofibrant. The dg-nerve of the dg-functor $\text{-}\otimes_A M : \mathrm{dgMod}_A^{\mathrm{cf}} \to \mathrm{dgMod}_B^{\mathrm{cf}}$ is a colimit preserving functor $\mathrm{N}_{\mathrm{dg}}(\text{-} \otimes_A M) : \mathcal{D}(A) \to \mathcal{D}(B)$ between presentable ∞-categories.

To compute limits and colimits in functor categories, we can use bimodules.

Lemma 2.17. *There exists an exact functor between stable ∞-categories*

$$\mathcal{D}(A^{\mathrm{op}} \otimes_k B) \longrightarrow \mathrm{Fun}(\mathcal{D}(A), \mathcal{D}(B))$$

which maps a dg A-B-bimodule M to $\text{-} \otimes_A^L M$.

Proof. The proof uses the technology of k-linear ∞-categories, which we have not yet introduced. Using [Lur17, 4.8.4.1], one can show that there exists an equivalence of ∞-categories

$$\mathcal{D}(A^{\mathrm{op}} \otimes_k B) \simeq \mathrm{RMod}_{A^{\mathrm{op}}\otimes_k B}(\mathrm{RMod}_k) \simeq \mathrm{Lin}_k^L(\mathrm{RMod}_A, \mathrm{RMod}_B) \simeq \mathrm{Lin}_k^L(\mathcal{D}(A), \mathcal{D}(B))\,,$$

where $\mathrm{Lin}_k^L(\mathcal{D}(A), \mathcal{D}(B))$ denotes the ∞-category of k-linear functors. Composing with the exact forgetful functors $\mathrm{Lin}_k^L(\mathcal{D}(A), \mathcal{D}(B)) \to \mathrm{Fun}^L(\mathcal{D}(A), \mathcal{D}(B)) \to \mathrm{Fun}^L(\mathcal{D}(A), \mathcal{D}(B))$ yields the result. □

(Co)limits in $\mathcal{D}(A)$ via homotopy (co)limits

Let A be a dg-category. We can compute limits and colimits in $\mathcal{D}(A)$ as (enriched) homotopy limits and colimits in the model category dgMod_A with the projective model structure. This essentially follows from [Lur09, 4.2.4.1]. To keep things non-technical, we only record how to compute fibers and cofibers.

Lemma 2.18. *Let $\alpha : a \to b$ be a morphism in $\mathrm{dgMod}_A^{\mathrm{cf}}$. Then there exist fiber and cofiber sequences in $\mathcal{D}(A)$*

$$\begin{tikzcd} \mathrm{cone}(\alpha)[-1] \arrow[r] \arrow[d, "\beta"] & 0 \arrow[d] \\ a \arrow[r, "\alpha"] & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d] & b \arrow[d, "\gamma"] \\ 0 \arrow[r] & \mathrm{cone}(\alpha) \end{tikzcd}$$

where $\mathrm{cone}(\alpha)$ *is the dg A-module with underlying chain complex* $\left(a[1] \oplus b, d = \begin{pmatrix} -d_a & -\alpha \\ 0 & d_b \end{pmatrix}\right)$. *The morphisms β and γ are the apparent cone maps.*

Proof. This follows from the comparison of homotopy (co)limits with (co)limits in the underlying ∞-categories and the fact, that the homotopy pushout or pullback along 0 is the cone or shifted cone. □

Lecture 3: Spherical adjunctions

Recall from Theorem 1.12, that there exists an equivalence between the abelian category $\mathrm{Perv}(\mathbf{D})$ of perverse sheaves on the once-punctured disc and the abelian category of diagrams of finite dimensional vector spaces

$$f : V \longleftrightarrow N : g \tag{5}$$

satisfying that $fg - \mathrm{id}_N$ and $gf - \mathrm{id}_V$ are equivalences. In this lecture, we discuss a categorification of this description of perverse sheaves on a disc in terms of so called spherical adjunctions. To obtain the categorifications, we apply the typical rules of categorification from Table 2.

algebraic concept	categorification
vector space	stable ∞-category
linear map	exact functor
difference	fiber/cofiber

Table 2: Typical rules for categorification.

To categorify the linear maps (5), we are thus asking for a pair of exact functors $F : \mathcal{V} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{V}$ between stable ∞-categories. To categorify the difference, we need to have a natural transformation $\mathrm{id}_{\mathcal{V}} \to GF$ (meaning a morphism in the stable ∞-category $\mathrm{Fun}(\mathcal{V}, \mathcal{V})$ of endofunctors) of which we can take the cofiber. Similarly, we are asking for a natural transformation $FG \to \mathrm{id}_{\mathcal{N}}$ of which we can take the fiber in the stable ∞-category $\mathrm{Fun}(\mathcal{N}, \mathcal{N})$. What we are looking for is the following:

Definition 3.1. Let $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ be a pair of functors of 1-categories. We call this pair an adjunction if there exist natural transformations $\mathrm{u} : \mathrm{id}_{\mathcal{C}} \to GF$ and $\mathrm{cu} : FG \to \mathrm{id}_{\mathcal{D}}$, satisfying that the following diagrams commute.

$$\begin{tikzcd} F \arrow[r, "F\,\mathrm{u}"] \arrow[dr, "\mathrm{id}"'] & FGF \arrow[d, "\mathrm{cu}\,F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\mathrm{u}\,G"] \arrow[dr, "\mathrm{id}"'] & GFG \arrow[d, "G\,\mathrm{cu}"] \\ & G \end{tikzcd} \tag{6}$$

We also call F left adjoint to G and G right adjoint to F. We write $F \dashv G$. The natural transformation u is called the unit of the adjunction and the natural transformation cu is called the counit of the adjunction.

To define an adjunction of functors between ∞-categories, one would start by asking for a unit and counit. There then need to exist choices of 2-simplicies rendering the diagrams (6) commutative. Higher categorical principles tell us that we should continue asking for 3-simplicies, which show that these 2-simplicies are in some sense compatible, and so on. Keeping track of all that data is rather challenging, but there are different approaches to avoid this problem. The original approach taken in [Lur09] encodes an adjunction as a fibration with lifting properties, which allows to recover unit, counit and all desired compatibility relations. An equivalent, more recent, and in practice easier to use approach, pioneered by Riehl-Verity, is to only require the existence of adjunction data up to the triangle identities. Surprisingly, this already gives the correct notion of adjunction. Definition 3.1 thus remains valid if $\mathcal{C}, \mathcal{D}$ are ∞-categories. For more discussions, see for instance [Lur23, Section 6.1] and [Cis19, Section 6.1].

We consider the following categorification of $gf - \mathrm{id}_V$ and $fg - \mathrm{id}_N$, called twist and cotwist.

Definition 3.2. Let $F : \mathcal{V} \leftrightarrow \mathcal{N} : G$ be an adjunction of stable ∞-categories.

- Let $\mathrm{u} : \mathrm{id}_{\mathcal{V}} \to GF$ be the unit transformation, considered as a morphism in the stable ∞-category $\mathrm{Fun}(\mathcal{V}, \mathcal{V})$. We define the twist functor $T_{\mathcal{V}} \in \mathrm{Fun}(\mathcal{V}, \mathcal{V})$ as the cofiber of u.
- Let $\mathrm{cu} : FG \to \mathrm{id}_{\mathcal{N}}$. We define the cotwist functor $T_{\mathcal{N}} \in \mathrm{Fun}(\mathcal{N}, \mathcal{N})$ as the fiber of cu.

The following definition is now a natural categorification of the description of perverse sheaves given in Theorem 1.12.

Definition 3.3. An adjunction of stable ∞-categories $F : \mathcal{V} \leftrightarrow \mathcal{N} : G$ is called spherical if both the twist $T_{\mathcal{V}}$ and the cotwist $T_{\mathcal{N}}$ are equivalences of ∞-categories.

In this case, the functors F and G are also called spherical functors.

The notion of a spherical functor is due to Anno-Logvinenko [AL17]. It is a generalization of the notion of a spherical object, introduced by Seidel-Thomas [ST01].

Example 3.4. An important example of a spherical adjunction is the following trivial example. Let $\mathcal{N}$ be a stable ∞-category and 0 be the stable ∞-category consisting only of a zero object (as a simplicial set, $0 = *$). Then the adjunction

$$0 : 0 \longleftrightarrow \mathcal{N} : 0$$

is spherical. The twist functor is the cofiber of the unit $u : \mathrm{id}_0 = 0 \to 0$ and thus the zero functor $0 : 0 \to 0$, which is an equivalence. The cotwist functor is the fiber of the counit $0 : 0 \to \mathrm{id}_{\mathcal{N}}$ and thus equivalent to the loop functor $[-1]$.

Decategorified, this example corresponds to the constant perverse sheaf on the disc, i.e. a the constant local system with value N.

Basic examples from algebra

To see the theory of spherical adjunctions in action, we compute a basic example. We fix a field k.

Example 3.5. Let $k[\epsilon] \coloneqq k[x]/(x^2)$ be the square-zero k-algebra. The morphism of algebras $\psi : k \to k[\epsilon]$ induces the adjunction

$$\psi_! = \text{-} \otimes_k^L k[\epsilon] : \mathcal{D}(k) \longleftrightarrow \mathcal{D}(k[\epsilon]) : \psi^* = \mathrm{RHom}_{k[\epsilon]}(k[\epsilon], \text{-})\,.$$

We compute the twist functor $T_{\mathcal{D}(k)} = \mathrm{cof}(\mathrm{id}_{\mathcal{D}(k)} \to \psi^*\psi_!)$. The functors $\psi_!$ and $\psi^*!$ arise as the dg-nerves of the functors $k[\epsilon] \otimes_k \text{-} : \mathrm{dgMod}_k^{\mathrm{cf}} \to \mathrm{dgMod}_{k[\epsilon]}$ and $\mathrm{Hom}_{k[\epsilon]}(k[\epsilon], \text{-}) : \mathrm{dgMod}_{k[\epsilon]}^{\mathrm{cf}} \to \mathrm{dgMod}_k^{\mathrm{cf}}$. We note that it follows from the fact that every dg k-module is cofibrant, that $\mathrm{Hom}_{k[\epsilon]}(k[\epsilon], \text{-})$ really preserves cofibrant modules. Using that the dg-nerve is functorial, we find that $\psi^*\psi_!$ is equivalent to the dg-nerve of the dg-functor

$$\mathrm{Hom}_{k[\epsilon]}(k[\epsilon], \text{-} \otimes_k k[\epsilon]) \simeq \text{-} \otimes_k k[\epsilon] : \mathrm{dgMod}_k \to \mathrm{dgMod}_k\,.$$

The morphism $\psi : k \to k[\epsilon]$ in dgMod_k induces a natural transformation $\text{-}\otimes_k \psi : \text{-}\otimes_k k \to \text{-}\otimes_k k[\epsilon]$. Applying the dg-nerve yields, up to equivalence, the unit transformation $\mathrm{u} : \mathrm{id}_{\mathcal{D}(k)} \to \psi^*\psi_!$. By Lemmas 2.17 and 2.18, we thus find that $\mathrm{cof}(\mathrm{u}) \simeq \text{-} \otimes_k \mathrm{cone}(\psi)$. One can find a quasi-isomorphism $\mathrm{cone}(\psi) \simeq k$, which shows that $T_{\mathcal{D}(k)} \simeq \mathrm{id}_{\mathcal{D}(k)}$. The twist functor $T_{\mathcal{D}(k)}$ is thus an equivalence.

Similarly, one can compute that the cotwist functor of $\psi_! \dashv \psi^*$ is also is equivalent to the identity functor. The adjunction $\psi_! \dashv \psi^*$ is hence spherical.

Exercise 4. Let $k[x]$ be the graded polynomial algebra with generator x in degree $|x| \in \mathbb{Z}$. The morphism of dg-algebras $\psi : k[x] \xrightarrow{x \mapsto 0} k$ induces the adjunction

$$\psi_! = \text{-} \otimes_{k[x]}^L k : \mathcal{D}(k[x]) \longleftrightarrow \mathcal{D}(k) : \psi^* = \mathrm{RHom}_k(k, \text{-})\,.$$

Show that the adjunction $\psi_! \dashv \psi^*$ is spherical.

Hints: Find a cofibrant replacement (=projective resolution) X of $k \in \mathrm{dgMod}_{k[x]}$. You may assume that ψ^* is equivalent to $\text{-} \otimes_k^L X$.

Figuring out the correct signs makes the computation more challenging. If you get stuck, try to first solve the case $|x| = 0$, where the signs are not so difficult. If you are still stuck, you can find the computation of the twist functor in the proof of Prop. 5.7 in [Chr22a].

The 2/4 property of spherical adjunctions

The 2/4 property of spherical adjunctions gives a number of equivalent criteria for an adjunction to be spherical. It originally appears in [AL17], for a proof in the setting of stable ∞-categories, we refer to [Chr22b]. We record the following special case of the 2/4 property.

Theorem 3.6 (Special case of the 2/4 property)**.** *Let $F : \mathcal{V} \leftrightarrow \mathcal{N} : G$ be an adjunction of stable ∞-categories. The adjunction $F \dashv G$ is spherical if and only if the following three hold.*

- *The functor F admits a left adjoint E.*
- *The twist functor $T_{\mathcal{V}}$ of $F \dashv G$ is an equivalence with inverse the cotwist functor of $E \dashv G$.*
- *Let u be the unit of $F \dashv G$ and u' the unit of $E \dashv F$. The natural transformation $G \to T_{\mathcal{V}}E$ contained in the diagram in the stable ∞-category $\mathrm{Fun}(\mathcal{N}, \mathcal{V})$*

$$\begin{tikzcd} & E \arrow[r] \arrow[d, "uE"'] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ G \arrow[r, "Gu'"] & GFE \arrow[r] & T_{\mathcal{V}}E \end{tikzcd}$$

is a natural equivalence.

Theorem 3.6 has the following nice corollary.

Corollary 3.7. *Let $F : \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction. Then there is an infinite sequence of adjunctions*

$$\cdots \dashv T_{\mathcal{V}}^{-1} G \dashv F \dashv G \dashv F T_{\mathcal{V}}^{-1} \dashv T_{\mathcal{V}} G \dashv \ldots$$

Furthermore, every adjunction above is spherical.

Examples from algebraic topology

Let $\mathcal{D}$ be any stable ∞-category and denote for $n \geq 0$ by S^n the singular set of the topological n-sphere. Consider the unique morphism of simplicial sets $f : S^n \to *$. Let $f^* : \mathcal{D} \simeq \mathrm{Fun}(*, \mathcal{D}) \to \mathrm{Fun}(S^n, \mathcal{D})$ be the pullback functor along f. Recall from lecture 2, that the functor f^* admits both left and right adjoints, $f_! = \mathrm{colim}$ and $f_* = \lim$.

Proposition 3.8 ([KS14] for the case $\mathcal{D} = \mathcal{D}^b(k)$ and [Chr22b, Prop. 3.9] for the general case)**.**

The adjunction $f^* \dashv f_*$ is spherical. The twist functor $T_{\mathcal{D}}$ of $f^* \dashv f_*$ is equivalent to the n-fold loop functor $[-n]$.

The functor f_* should be thought of as taking the cohomology of the n-sphere with values in a $\mathcal{D}$-valued local system. Similarly, the functor $f_!$ should be thought of as taking homology. Indeed, if we suppose that $\mathcal{D} = \mathcal{D}(k)$, one can show, that

$$f_* f^*(k) \simeq H^*(S^n, k) \simeq k \oplus k[-n]$$

is the singular cohomology of the n-sphere with coefficients in the constant local system $f^*(k)$ with value k. More generally, one finds $f_* f^* \simeq \mathrm{id}_{\mathcal{D}} \oplus [-n]$ and the unit of $f^* \dashv f_*$ is given by the inclusion $\mathrm{id} \hookrightarrow \mathrm{id} \oplus [-n] \simeq f_* f^*$. This shows that the twist $T_{\mathcal{D}}$ is equivalent to $[-n]$, which is an equivalence.

To show that the adjunction $f^* \dashv f_*$ is spherical, one could try to compute the cotwist. For a general choice of $\mathcal{D}$, this is probably not so easy. Instead, we apply Theorem 3.6. We again assume that $\mathcal{D} = \mathcal{D}(k)$. Homology and cohomology are related via Poincaré duality $H^*(S^n) \simeq H_*(S^n)[-n]$. This is captured by the fact, that the natural transformation $f_* \to T_{\mathcal{D}} f_! \simeq [-n] f_!$ from Theorem 3.6 is a natural equivalence.

Remark 3.9. Proposition 3.8 admits the following generalization: Let $f : X \to Y$ be a Kan fibration between Kan complexes, whose fiber is equivalent to the n-sphere for some $n \in \mathbb{N}$. Such a fibration is called a spherical fibration. Then for every stable ∞-category $\mathcal{D}$, the adjunction

$$f^* : \mathrm{Fun}(Y, \mathcal{D}) \longleftrightarrow \mathrm{Fun}(X, \mathcal{D}) : f_*$$

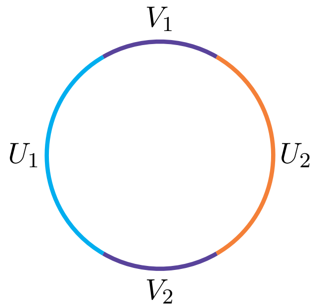
is spherical.

An example of a spherical fibration is the Hopf fibration $S^3 \to S^2$.

Sheaf interpretation of cotwist

Recall from lecture 1, that given a perverse sheaf F on the once-punctured disc D, the restriction to the complement $\mathbf{D} \backslash \{p\}$ of the puncture p defines a local system L. Since $\mathbf{D} \backslash \{p\} \simeq S^1$ is homotopy equivalent to the circle, we can consider L as a local system on S^1. Let N be its stalk.

Going once counterclockwise around the circle, there is an associated monodromy automorphism $N \to N$. It is defined as follows. Let $U_1, U_2 \subset S^1$ be two connected, contractible open sets, such that $U_1 \cup U_2 = S^1$. Let $V_1, V_2 \subset U_1, U_2$ be the two connected components of the intersection.



The monodromy can be defined as the automorphism

$$N \simeq H^*(U_1; L) \xrightarrow{\text{res}} H^*(V_1; L) \xrightarrow{\text{res}^{-1}} H^*(U_2; L) \xrightarrow{\text{res}} H^*(V_2; L) \xrightarrow{\text{res}^{-1}} H^*(U_1; L) \simeq N$$

using that the restriction maps above are all invertible.

Let $f : V \leftrightarrow N : g$ be the diagram in the abelian category $\mathcal{A}_1$ associated to $F \in \mathrm{Perv}(\mathbf{D})$. Then one can show that the monodromy equivalence is equivalent to $fg - \mathrm{id}_N$. We should thus think of the cotwist autoequivalence of a spherical adjunction as a monodromy.

Example 3.10. Consider the trivial spherical adjunction $0 \leftrightarrow \mathcal{N}$. This example corresponds to the categorification of the constant perverse sheaf on the disc. The monodromy should thus be trivial, i.e. equivalent to the identity. The cotwist is however given by $[-1]$. The definition of the cotwist as a cofiber rather than a fiber is simply a convention; the suspension of the cotwist is the correct monodromy.

Rotational symmetry

Let $F \in \mathrm{Perv}(\mathbf{D})$ again be a perverse sheaf on the once-punctured disc. The graph Γ_n from Figure 2 has an apparent rotational symmetry, given by rotating by $\frac{2\pi}{n}$. Consider the diagram in $\mathcal{A}_n$

$$(r_i : V^n \longleftrightarrow N_i : s_i)_{1 \le i \le n}$$

associated to F. The rotational symmetry is realized in terms of the stalks N_i by the equivalence $r_i s_{i+1} : N_{i+1} \to N_i$. Composing all these equivalences

$$N_1 \xrightarrow{r_n s_1} N_n \xrightarrow{r_{n-1} s_n} \dots \xrightarrow{r_2 s_3} N_2 \xrightarrow{r_1 s_2} N_1$$

we again obtain the monodromy of the local system on $\mathbf{D}\backslash\{p\}$ given by the restriction of F.

It is interesting to note, that these equivalences can be organized into an object of $\mathcal{A}_1$. Consider the linear maps

$$R \coloneqq \bigoplus_{i=1}^{n} r_i : V^n \longleftrightarrow \bigoplus_{i=1}^{n} N_i : \bigoplus_{i=1}^{n} s_i \eqqcolon S$$

Since $r_i s_j = 0$ if $i \neq j, j-1$ and $r_j s_j = \mathrm{id}_{N_j}$, it follows that $RS - \mathrm{id}_{\bigoplus_{i=1}^n N_i} = \bigoplus_{i=1}^n r_i s_{i+1}$ is invertible. One can also show that $SR - \mathrm{id}_{V^n}$ is invertible.

We will make use of these facts in the next lecture, when we describe the categorification of the objects in $\mathcal{A}_n$.

Lecture 4: Perverse schobers via ribbon graphs

We wish to categorify the notion of a perverse sheaf. Since we cannot (yet?) make sense of the derived category of the ∞-category of complexes of stable ∞-categories, a direct categorification of the definition of a perverse sheaf in terms of objects in the heart of some perverse t-structure is currently not possible. We managed to get around this problem for perverse sheaves on a disc, by using the linear algebra description of perverse sheaves on the disc in terms of diagrams of vector spaces $r : V^1 \leftrightarrow N : s$, satisfying that $rs - \mathrm{id}_N, sr - \mathrm{id}_{V^1}$ are invertible. The categorification is a spherical adjunction. We can thus give the following definition of a perverse schober on the once-punctured disc as proposed by Kapranov-Schechtman [KS14]. We denote by $\mathrm{St} \subset \mathrm{Cat}_\infty$ the subcategory of stable ∞-categories and exact functors.

Definition 4.1. Let Γ_1 be the ribbon graph with a single vertex and a single edge. A Γ_1-parametrized perverse schober $\mathcal{F}$ consists of a spherical adjunction

$$F\colon \mathcal{V} \longleftrightarrow \mathcal{N} : G\,.$$

Note that we can recover the right adjoint G from F, so that we may consider a Γ_1-parametrized perverse schober as an object in the functor category $\mathrm{Fun}(\mathrm{Exit}(\Gamma_1), \mathrm{St})$.

In this lecture, we apply this approach of categorification to perverse sheaves on arbitrary marked surfaces with nonempty boundary. We noted in lecture 1, that perverse sheaves on a such a surface can be described in terms of functors

$$\mathrm{Exit}(\Gamma) \amalg_{\mathrm{ob}(\mathrm{Exit}(\Gamma))} \mathrm{Exit}(\Gamma)^{\mathrm{op}} \to \mathrm{Vect}_k^{\mathrm{fin}}$$

from a bidirectional version of the exit path category of a spanning graph Γ of $\mathbf{S}$. These functors need to locally be of the forms described in Theorem 1.12 and Theorem 1.13. We will thus begin by describing the categorification of the description from Theorem 1.13, first abstractly, and in terms of an explicit and computational model defined in terms of sections of a Grothendieck construction.

Parametrized perverse schobers on a disc (abstractly)

Let Γ_n be the ribbon graph with a single vertex and n incident (external) edges.

Definition 4.2. Let $n \geq 2$. A Γ_n-parametrized perverse schober consists of a collection of adjunctions

$$(F_i\colon \mathcal{V}^n \longleftrightarrow \mathcal{N}_i : G_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$$

between stable ∞-categories satisfying that

(a) G_i is fully faithful, i.e. $F_iG_i \simeq \mathrm{id}_{\mathcal{N}_i}$ via the counit,

(b) $F_i \circ G_{i+1}$ is an equivalence of ∞-categories,

(c) $F_i \circ G_j \simeq 0$ if $j \neq i, i+1$,

(d) G_i admits a right adjoint $\mathrm{radj}(G_i)$ and F_i admits a left adjoint $\mathrm{ladj}(F_i)$ and

(e) $\mathrm{fib}(\mathrm{radj}(G_{i+1})) = \mathrm{fib}(F_i)$ as full subcategories of $\mathcal{V}^n$, where the fiber fib refers to the pullback in St along the zero stable ∞-category.

Since we can recover the right adjoints from the left adjoints, we may also consider the collection of functors $(F_i\colon \mathcal{V}^n \to \mathcal{N}_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ as Γ_n-parametrized perverse schober. Such a collection of functors can equivalently be encoded as a functor in $\mathrm{Fun}(\mathrm{Exit}(\Gamma_n), \mathrm{St})$.

Remark 4.3. Conditions (a)-(c) in part (2) of Definition 4.2 are apparent categorifications of the conditions from Theorem 1.13. Conditions (d) and (e) can be justified, by showing that the data of Γ_n-parametrized perverse schobers are equivalent for different n. While this is fully explained in [CHQ23] or [Chr23] and we give a partial account in Theorem 4.13.

We note also that condition (e) is equivalent to the condition that the images $\mathrm{Im}(G_{i+1})$ and $\mathrm{Im}(\mathrm{ladj}(F_i))$ agree. Since G_{i+1} and $\mathrm{ladj}(F_i)$ are fully faithful, it follows that $\mathrm{ladj}(F_i)$ differs from G_{i+1} by composition with the equivalence $(G_{i+1})^{-1} \circ \mathrm{ladj}(F_i) \colon \mathcal{N}_i \simeq \mathcal{N}_{i+1}$.

Given a spherical functor F, there exists for each $n \geq 2$ a corresponding Γ_n-parametrized perverse schober, and vice versa. To define the value $\mathcal{V}^n_F$ of the perverse schober at the vertex of Γ_n, we need to introduce a very versatile and widely applicable ∞-categorical constructions, called the Grothendieck construction. Besides its use in this lecture, it will also play a central role in the description of the ∞-categories of global sections in the coming lectures.

The Grothendieck construction

Let Z be a 1-category and $f : Z \to \mathrm{Set}_\Delta$ a functor taking values in ∞-categories. The (covariant) Grothendieck construction consists of an ∞-category $\Gamma(f)$ together with a functor $p : \Gamma(f) \to N(Z)$. For the explicit definition as as simplicial sets, we refer to [Lur09, 3.2.5.2, 3.2.5.4], where the Grothendieck construction is called the nerve of Z relative f.

An object (z, x) in $\Gamma(f)$ consists of an object $z \in Z$ and an object x in the ∞-category $f(x)$. A morphism $(z, x) \to (z', x')$ in $\Gamma(f)$ consists of a morphism $\alpha : z \to z'$ in Z and a morphism $f(\alpha)(x) \to x'$ in the ∞-category $f(z')$. The functor p projects an object (z, x) to z. The fiber of p at $z \in Z$ is thus given by the ∞-category $f(z)$, which is in particular a full subcategory of $\Gamma(f)$.

In this lecture, we are interested in the case that Z the poset 1-category $[n]$ consisting of the objects $0, \dots, n$ with the apparent order. The nerve is given by $N([n]) \simeq \Delta^n$. The functor $f : [n] \to \mathrm{Set}_\Delta$ in this case encodes n many composable functors between ∞-categories:

$$f(0) \xrightarrow{f(0\to 1)} f(1) \xrightarrow{f(1\to 2)} \dots \xrightarrow{f(n-1\to n)} f(n)$$

In the special case that $n = 1$, where $f : [1] \to \mathrm{Set}_\Delta$ describes a functor $F : f(0) \to f(1)$ between ∞-categories, we also write $\Gamma(F) \coloneqq \Gamma(f)$.

Given a functor $f : Z \to \mathrm{Set}_\Delta$, even if it takes values in stable ∞-categories, its Grothendieck construction $\Gamma(f)$ will, unless $Z = *$, not be a stable ∞-category. For example if $Z = [1]$, there are no morphisms, and hence no zero morphisms, from any object $(1, x)$ to any object $(0, x')$, because there is no morphism $1 \to 0$ in $[1]$. Crucially, the ∞-category of sections of $p : \Gamma(f) \to Z$, see Definition 4.5, is however stable.

Lemma 4.4. *Let $f : Z \to \mathrm{Set}_\Delta$ being a functor taking values in stable ∞-categories. The ∞-category of sections $\mathrm{Fun}_{N(Z)}(N(Z), \Gamma(f))$ is stable.*

Proof. Assume that f takes values in presentable ∞-categories and colimit preserving functors. The Grothendieck construction $p : \Gamma(f) \to N(Z)$ is then both a coCartesian fibration and a Cartesian fibration, see [Lur09, 3.2.5.21, 5.2.2.5]. It thus follows from [Lur09, 5.1.2.2] that limits and colimits in $\mathrm{Fun}_{N(Z)}(N(Z), \Gamma(f))$ are computed pointwise in the stable fibers of p. This shows that $\mathrm{Fun}_{N(Z)}(N(Z), \Gamma(f))$ is also stable.

If f does not take values in presentable ∞-categories, we can compose with f with the Ind-completion functor, obtaining a functor $\mathrm{Ind} \circ f : Z \to \mathrm{Set}_\Delta$. The ∞-category $\mathrm{Fun}_{N(Z)}(N(Z), \Gamma(f))$ is a full subcategory of $\mathrm{Fun}_{N(Z)}(N(Z), \Gamma(\mathrm{Ind} \circ f))$ which is closed under the formation of finite limits and colimits in $\mathrm{Fun}_{N(Z)}(N(Z), \Gamma(\mathrm{Ind} \circ f))$, and thus itself stable. □

Definition 4.5. Let $p : \mathcal{A} \to \mathcal{B}$ be a functor between ∞-categories (such as the Grothendieck construction). We define the ∞-category of sections of p as the pullback in Set_Δ

$$\mathrm{Fun}_{\mathcal{B}}(\mathcal{B}, \mathcal{A}) = \mathrm{Fun}(\mathcal{B}, \mathcal{A}) \times_{\mathrm{Fun}(\mathcal{B},\mathcal{B})} \{\mathrm{id}_{\mathcal{B}}\}\,.$$

The objects in the ∞-category of sections $\mathrm{Fun}_{\mathcal{B}}(\mathcal{B}, \mathcal{A})$ are functors $D : \mathcal{B} \to \mathcal{A}$ (i.e. $\mathcal{B}$ indexed diagrams), satisfying that the composite functor $p \circ D$ is the identity functor on $\mathcal{B}$.

Suppose again that $Z = [n]$ and that f takes values in stable ∞-categories. The the stable ∞-category $\mathrm{Fun}_{\Delta^n}(\Delta^n, \Gamma(f))$ has as objects diagrams of the form

$$x_0 \xrightarrow{\alpha_{0,1}} x_1 \xrightarrow{\alpha_{1,2}} \dots \xrightarrow{\alpha_{n-1,n}} x_n$$

where $x_i \in f(i) \subset \Gamma(f)$ and the morphisms $\alpha_{i,i+1} : x_i \to x_{i+1}$ in $\Gamma(f)$ encode morphisms $f(i \to i+1)(x_i) \to x_{i+1}$ in $f(i+1)$. Morphisms in the ∞-category of sections are natural transformations between diagrams. In the special case $Z = [n]$, morphisms in $\mathrm{Fun}_{\Delta^{n-1}}(\Delta^{n-1}, \Gamma(f))$ boil down to commutative diagrams of the form

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & x_1 \arrow[r] \arrow[d] & \dots \arrow[r] & x_n \arrow[d] \\ y_0 \arrow[r] & y_1 \arrow[r] & \dots \arrow[r] & y_n \end{tikzcd}$$

in $\Gamma(F)$ with $x_i, y_i \in f(i)$.

Example 4.6. Let $f : [n] \to \mathrm{Set}_\Delta$ be the constant diagram with value a stable ∞-category $\mathcal{N}$. This diagram corresponds to the composable functors

$$\mathcal{N} \xrightarrow{\mathrm{id}_{\mathcal{N}}} \dots \xrightarrow{\mathrm{id}_{\mathcal{N}}} \mathcal{N}\,.$$

One can show that there is an equivalence of ∞-categories

$$\mathrm{Fun}_{\Delta^n}(\Delta^n, \Gamma(f)) \simeq \mathrm{Fun}(\Delta^n, \mathcal{N})\,.$$

If $\mathcal{N} = \mathcal{D}(k)$, then there exists furthermore an equivalence of ∞-categories

$$\mathrm{Fun}(\Delta^n, \mathcal{D}(k)) \simeq \mathcal{D}(kA_{n+1})$$

where kA_{n+1} is the path algebra of the quiver

$$A_{n+1} = \cdot \to \cdot \to \cdots \to \cdot$$

with $n+1$ vertices. Under this equivalence, a representation $a_0 \to a_1 \to \cdots \to a_n$ of the A_{n+1} quiver, where $a_i \in \mathrm{Vect}_k \subset \mathcal{D}(k)$, is mapped to this representation considered as a section of $\Gamma(f)$.

Example 4.7. Let A, B be dg-algebras and M a cofibrant dg A-B-bimodule. Consider the associated functor $F = \text{-} \otimes^L_A M : \mathcal{D}(A) \to \mathcal{D}(B)$. The stable ∞-category $\mathrm{Fun}_{\Delta^1}(\Delta^1, \Gamma(F))$ is equivalent to the derived ∞-category of the upper triangular dg-algebra $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. See [Chr22a, Lemma 2.29, Prop. 2.39] for a proof. This upper triangular dg-algebra is also sometimes called the gluing of A and B along M.

Parametrized perverse schobers on a disc (concretely)

We now have the tools to define a Γ_n-parametrized perverse schober, given any $n \geq 2$ and spherical functor F.

Definition 4.8. Let $F : \mathcal{V} \to \mathcal{N}$ be a spherical functor. Consider the diagram $f : [n-1] \to \mathrm{Set}_\Delta$ corresponding to the composable functors

$$\mathcal{V} \xrightarrow{F} \underbrace{\mathcal{N} \xrightarrow{\mathrm{id}_\mathcal{N}} \dots \xrightarrow{\mathrm{id}_\mathcal{N}} \mathcal{N}}_{n-1 \text{ many}} .$$

We define $\mathcal{V}^n_F = \mathrm{Fun}_{\Delta^{n-1}}(\Delta^{n-1}, \Gamma(f))$.

Note that for $n = 1$, we have $\mathcal{V}^1_F \simeq \mathcal{V}$.

Definition 4.9. Let $F : \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction and $n \geq 1$. We define functors $\varrho_i : \mathcal{V}^n_F \to \mathcal{N}$ and $\varsigma_i : \mathcal{N} \to \mathcal{V}^n_F$ for $1 \leq i \leq n$ as follows.

1. If $n = 1$, we set $\varrho_1 = F$ and $\varsigma_1 = G$.

2. Let $n \geq 2$. We set $\varrho_1 = \mathrm{ev}_{n-1}$ to be the functor which evaluates a section at the vertex $n-1 \in \Delta^{n-1}$. It is given on objects by

$$\varrho_1 : (a \to b_1 \to \dots \to b_{n-1}) \mapsto b_{n-1} .$$

We set for $2 \leq i \leq n-1$ the functor $\varrho_i = \mathrm{fib}_{n-i-1,n-i}[i-1]$ to be the suspension of the composite of the pullback along $\Delta^1 \simeq \Delta^{\{n-i-1,n-i\}} \hookrightarrow \Delta^{n-1}$ with the fiber functor. It is given on object by

$$\varrho_i : \left(a \to b_1 \xrightarrow{\alpha_{1,2}} \dots \xrightarrow{\alpha_{n-2,n-1}} b_{n-1}\right) \mapsto \mathrm{fib}(\alpha_{n-i-1,n-i})[i-1] .$$

We set $\varrho_n = \mathrm{rfib}_{0,1}[n-1]$ to be the suspension of the composite of the pullback along $\Delta^1 \simeq \Delta^{\{0,1\}} \hookrightarrow \Delta^{n-1}$ with the relative fiber functor that assigns to an object $a \to b \in \mathrm{Fun}_{\Delta^1}(\Delta^1, \Gamma(F))$ the object $\mathrm{fib}(F(a) \to b) \in \mathcal{N}$. One can define the relative fiber functor as the left adjoint of the inclusion $\mathcal{N} \subset \mathcal{V}^2_F$ composed with $[-1]$. On objects, it is given by

$$\varrho_n : \left(a \xrightarrow{\alpha} b_1 \to \dots \to b_{n-1}\right) \mapsto \mathrm{fib}(F(a) \to b)[n-1] ,$$

where $F(a) \to b$ is the morphism in $\mathcal{N}$ encoded by the morphism α in $\Gamma(F)$.

3. If $n \geq 2$, we set ς_i to be the right adjoint of ϱ_i. The functor ς_1 is given on objects by

$$\varsigma_1 : b \mapsto \left(G(b) \xrightarrow{*} b \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} b\right) ,$$

where the morphism $G(b) \xrightarrow{*} b$ in $\Gamma(F)$ encodes the counit map $FG(b) \to b$. For $2 \leq i \leq n$, the functor ς_i is given on objects by

$$\varsigma_i : b \mapsto \left(0 \to \dots \to \underbrace{b[-i+2]}_{n-i+1\text{-th}} \to \dots \to 0 \right) .$$

Remark 4.10. It is fairly easy to show that there is a sequence of adjunctions

$$\varrho_n \dashv \varsigma_n \dashv \varrho_{n-1} \dashv \dots \dashv \varsigma_2 \dashv \varrho_1 \dashv \varsigma_1 . \tag{7}$$

With some more work, one can also show that there is a further adjunction

$$\varsigma_1 T_\mathcal{N}^{-1}[1-n] \dashv \varrho_n$$

where $T_\mathcal{N}$ is the cotwist functor of $F \dashv G$.

Remark 4.11. The functors $\varsigma_i : \mathcal{N} \to \mathcal{V}^n_F$ are fully faithful. It follows that the counits $\varrho_i\varsigma_i \to \mathrm{id}_{\mathcal{N}}$ and units $\mathrm{id}_{\mathcal{N}} \to \varrho_i\varsigma_{i+1}$ (or $T_{\mathcal{N}}[n-1] \to \varrho_n\varsigma_1$) are natural equivalences. One easily checks that $\varrho_j \circ \varsigma_i$ vanishes if $j \neq i, i-1$.

Definition 4.12. For $n \geq 1$, let Γ_n be the spanning graph of the n-gon depicted in Figure 2. Choosing any of its edges, we can label the edges of Γ_n by $e_1, \ldots, e_n$, compatible with their given cyclic order. Let $F : \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction. We define the functor $\mathcal{F}_n(F) : \mathrm{Exit}(\Gamma_n) \to \mathrm{St}$ by $\mathcal{F}_n(F)(v \to e_i) = \varrho_i$.

Theorem 4.13 ([CHQ23, Chr23])**.**

(i) Let F be a spherical functor. Then functor $\mathcal{F}_n(F) \in \mathrm{Fun}(\mathrm{Exit}(\Gamma_n), \mathrm{St})$ is a Γ_n-parametrized perverse schober.

(ii) Let $\mathcal{F}\colon \mathrm{Exit}(\Gamma_n) \to \mathrm{St}$ be a Γ_n-parametrized perverse schober. Then there exists a spherical functor F and an equivalence $\mathcal{F} \simeq \mathcal{F}_n(F)$ in $\mathrm{Fun}(\mathrm{Exit}(\Gamma_n, \mathrm{St})$.

Remark 4.14. The definition of the ∞-category $\mathcal{V}^n_F$ and the functors ϱ_i should be credited to Dyckerhoff's paper on the categorified Dold-Kan correspondence [Dyc21], where the relation to perverse schobers is already anticipated. There, the ∞-category $\mathcal{V}^n_F$ arise, up to equivalence, by applying the categorified Dold-Kan correspondence to the spherical functor F, considered as a chain complex of stable ∞-categories in degrees $0, 1$. This boils down to a description of $\mathcal{V}^n_F$ and the ϱ_i in terms of Waldhausen's relative $S_\bullet$ construction.

Local rotational symmetry

The spanning graph from Figure 2 of the n-gon has a rotational symmetry (also called cyclic symmetry), given by rotation by $\frac{2\pi}{n}$. At the end of lecture 3, we described the action of this symmetry on the objects in $\mathcal{A}_n$ via some linear maps $R : V^n \leftrightarrow \bigoplus_{i=1}^n N_i : S$ in $\mathcal{A}_1$. In the following, we extend this rotational symmetry to Γ_n-parametrized perverse schobers. If the monodromy is non-trivial, a full rotation does not the diagrams to themselves. Because of that, one may also call this symmetry a paracyclic symmetry (instead of a cyclic symmetry).

Lemma 4.15 ([Chr22a, Lemma 3.8])**.** *Let $F : \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction. Then the adjunction*

$$R \coloneqq (\varrho_1, \ldots, \varrho_n) : \mathcal{V}^n_F \longleftrightarrow \mathcal{N}^{\times n} : (\varsigma_1, \ldots, \varsigma_n) \eqqcolon S \tag{8}$$

is also spherical.

Proposition 4.16. *Let $F : \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction with cotwist functor $T_{\mathcal{N}}$ and consider the twist functor $T_{\mathcal{V}^n_F}$ of the spherical adjunction (8). Then there exist equivalences of functors*

$$\varrho_i \circ T_{\mathcal{V}^n_F} \simeq \begin{cases} \varrho_{i+1} & \text{for } 1 \leq i \leq n-1 \\ T_{\mathcal{N}}[n-1] \circ \varrho_1 & \text{for } i = n \end{cases} \tag{9}$$

and

$$T^{-1}_{\mathcal{V}^n_F} \circ \varsigma_i \simeq \begin{cases} \varsigma_{i+1} & \text{for } 1 \leq i \leq n-1 \\ \varsigma_1 \circ T^{-1}_{\mathcal{N}}[1-n] & \text{for } i = n \end{cases} \tag{10}$$

Proof. It follows from Remark 4.10 that there exists an adjunction

$$(\varsigma_2, \ldots, \varsigma_n, \varsigma_1 T^{-1}_{\mathcal{N}}[1-n]) : \mathcal{N}^{\times n} \longleftrightarrow \mathcal{V}^n_F : (\varrho_1, \ldots, \varrho_n)\,.$$

It now follows from Corollary 3.7, that

$$T^{-1}_{\mathcal{V}^n_F} \circ (\varsigma_1, \varsigma_2, \ldots, \varsigma_n) \simeq (\varsigma_2, \ldots, \varsigma_n, \varsigma_1 T^{-1}_{\mathcal{N}}[1-n])\,,$$

showing (10). The equivalences (9) follow from passing to left adjoints. $\square$

Proposition 4.16 implies that if we choose a different total order of the edges of the spanning graph Γ_n of the n-gon in Definition 4.12, then the functor $\mathcal{F}_n(F)$ only changes up to an equivalence in $\mathrm{Fun}(\mathrm{Exit}(\Gamma_n), \mathrm{St})$.

Parametrized perverse schobers

Having now both good abstract models, see Definition 4.2, and good explicit models, see Definition 4.12, for the local description of perverse schobers, we turn to the global description of perverse schobers on surfaces using graphs, or more precisely, ribbon graphs.

Remark 4.17. A ribbon graph is a graph Γ equipped with, for each vertex v of Γ, the datum of a cyclic ordering of the (half-)edges incident to v.

Given a graph embedded in an oriented surface, it canonically inherits the structure of a ribbon graph, by choosing the cyclic ordering to be the counterclockwise ordering in the surface.

Let Γ be a ribbon graph and v a vertex of Γ of valency n. There is a canonical functor $i_v \colon \mathrm{Exit}(\Gamma_n) \to \mathrm{Exit}(\Gamma)$ which maps the unique vertex of Γ_n to v, which is injective on objects, and which respects the cyclic orders of the edges. If Γ has no loops incident to v, then i_v is a fully faithful functor.

Definition 4.18. Let Γ be a ribbon graph. A functor $\mathcal{F} : \mathrm{Exit}(\Gamma) \to \mathrm{St}$ is called a Γ-parametrized perverse schober if for each vertex v of Γ of valency n, the functor $\mathcal{F} \circ i_v$ defines a Γ_n-parametrized perverse schober. This is equivalent to the assertion, that, for each vertex v, there exists a spherical functor $F_v : \mathcal{V}_v \to \mathcal{N}$ and an equivalence

$$\mathcal{F} \circ i_v \simeq \mathcal{F}_n(F_v)$$

in $\mathrm{Fun}(\mathrm{Exit}(\Gamma)_n, \mathrm{St})$. If $\mathcal{V}_v \not\simeq 0$, we call v a singularity of $\mathcal{F}$.

Definition 4.19. The ∞-category of Γ-parametrized perverse schobers $\mathfrak{P}(\Gamma) \subset \mathrm{Fun}(\mathrm{Exit}(\Gamma), \mathrm{St})$ is the full subcategory spanned by perverse schobers.

Remark 4.20. Every Γ-parametrized perverse schober assigns to each edge of Γ the same stable ∞-category, up to equivalence (which was always denoted by $\mathcal{N}$ above). We call this ∞-category, well defined up to equivalence, the generic stalk of the perverse schober.

Lecture 5: Global sections of perverse schobers

Definition 5.1. Let $\mathcal{F} : \mathrm{Exit}(\Gamma) \to \mathrm{St}$ be a Γ-parametrized perverse schober. The ∞-category of global sections $\mathcal{H}(\Gamma, \mathcal{F})$ of $\mathcal{F}$ is defined as the limit of $\mathcal{F}$.

Remark 5.2. Since a Γ-parametrized perverse schober should be thought of as the constructible sheaf of sections with support in Γ of some hypothetical abstract perverse schober (we don't yet know how to define such a gadget), the ∞-category of global sections of a Γ-parametrized perverse schober should be considered the ∞-category of global sections with support on Γ of the abstract perverse schober.

We begin this lecture by discussing how limits of ∞-categories are computed.

Limits in ∞-categories of ∞-categories

We will be interested in the following ∞-categories of ∞-categories.

- Cat_∞, the ∞-category of ∞-categories.
- St, the subcategory of Cat_∞ of stable ∞-categories and exact functors.

- $\mathcal{P}r^L$, the subcategory of Cat_∞ of presentable ∞-categories and colimit preserving (=left adjoint) functors.
- $\mathcal{P}r^R$, the subcategory of Cat_∞ of presentable ∞-categories and limit preserving and accessible (=right adjoint) functors.
- $\mathcal{P}r^L_{\mathrm{St}} \subset \mathcal{P}r^L$ and $\mathcal{P}r^R_{\mathrm{St}} \subset \mathcal{P}r^R$, the full subcategories consisting of stable and presentable ∞-categories.

The forgetful functors can be organized in a commutative diagram as follows.

$$\begin{tikzcd}
\mathcal{P}r^L_{\mathrm{St}} \arrow[r] \arrow[drr] & \mathcal{P}r^L \arrow[r] & \mathrm{Cat}_\infty & \mathcal{P}r^R \arrow[l] & \mathcal{P}r^R_{\mathrm{St}} \arrow[l] \arrow[dll] \\
 & & \mathrm{St} \arrow[u] & &
\end{tikzcd}$$

All of the above functor preserve all limits. To compute colimits in $\mathcal{P}r^L$ and $\mathcal{P}r^R$, we can use the following result:

Theorem 5.3 ([Lur09, 5.5.3.4])**.** *There are inverse equivalences of ∞-categories*

$$\mathrm{radj} : \mathcal{P}r^L \longleftrightarrow \left(\mathcal{P}r^R\right)^{\mathrm{op}} : \mathrm{ladj}\,,$$

where radj *is the identity on objects and maps a functor to its right adjoint. Similarly,* ladj *maps a functor to its left adjoint.*

Corollary 5.4. *The colimit of a diagram $Z \to \mathcal{P}r^L$ is equivalent to the limit of the right adjoint diagram $Z^{\mathrm{op}} \to \mathcal{P}r^R$.*

The equivalence from Theorem 5.3 clearly restricts to an equivalence between $\mathcal{P}r^L_{\mathrm{St}}$ and $(\mathcal{P}r^R_{\mathrm{St}})^{\mathrm{op}}$. It follows that Corollary 5.4 also holds for limits and colimits in $\mathcal{P}r^L_{\mathrm{St}}$ and $\mathcal{P}r^R_{\mathrm{St}}$.

We may thus focus on the computation of limits in Cat_∞. Let Z be a 1-category and $f : Z \to \mathrm{Set}_\Delta$ a functor valued in ∞-categories. The functor f defines a functor $F : N(Z) \to \mathrm{Cat}_\infty$, whose limit can be described in terms of sections of the Grothendieck construction $p : \Gamma(f) \to N(Z)$.

Definition 5.5. A section $X : N(Z) \to \Gamma(f)$ of the Grothendieck construction p is called coCartesian if for each morphism $\alpha : i \to j$ in $N(Z)$, the morphism $X(\alpha) : X(i) \to X(j)$ in $\Gamma(f)$, with $X(i) \in f(i)$ and $X(j) \in f(j)$, encodes an equivalence $f(\alpha)(X(i)) \simeq X(j)$.

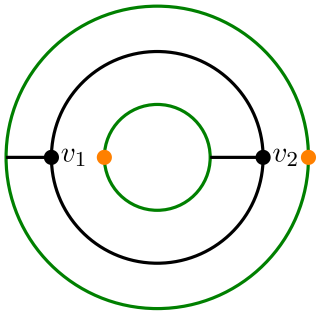
We remark, that one can check that such an $X(\alpha)$ defines a p-coCartesian edge in the sense of [Lur09, page 118 or the dual of Def. 2.4.1.1].

Theorem 5.6. *The limit of F is equivalent to the full subcategory of the ∞-category of sections $\mathrm{Fun}_{N(Z)}(N(Z), \Gamma(f))$ of the Grothendieck construction of f consisting of coCartesian sections.*

Proof. This appears as Corollary 7.4.1.10 in Lurie's Kerodon. Alternatively, a similar statement appears in [Lur09, 3.3.3.2] (warning: there are some typos). $\square$

Example: topological Fukaya category of an annulus = $\mathcal{D}(\mathrm{Coh}(\mathbb{P}^1))$

We consider the annulus with two marked points and the following spanning ribbon graph Γ:



(11)

We consider the following Γ-parametrized perverse schober $\mathcal{F}$.

$$\begin{tikzcd} & & \mathcal{D}(k) & \\ \mathcal{D}(k) & \operatorname{Fun}(\Delta^1, \mathcal{D}(k)) \arrow[l, "\varrho_2"] \arrow[ur, "\varrho_1"] \arrow[dr, "\varrho_3"'] & \mathcal{D}(k) & \operatorname{Fun}(\Delta^1, \mathcal{D}(k)) \arrow[l, "\varrho_2"] \arrow[ul, "\varrho_1"'] \arrow[dl, "\varrho_3"] \\ & & \mathcal{D}(k) & \end{tikzcd} \tag{12}$$

The schober $\mathcal{F}$ is locally constant, i.e. has no singularities, and the generic stalk is $\mathcal{D}(k)$, for k a commutative ring. Recall that $\varrho_1 = \mathrm{ev}_1, \varrho_2 = \mathrm{cof}$ and $\varrho_3 \simeq \mathrm{ev}_0[1]$. An arbitrary global section of $\mathcal{F}$, described as a coCartesian section of the Grothendieck construction, is therefore a coCartesian section of the form (up to equivalence):

$$\begin{tikzcd} & & a_1 & \\ \operatorname{cof}(\alpha) & \left(a_0 \xrightarrow{\alpha} a_1\right) \arrow[l] \arrow[ur] \arrow[dr] & \operatorname{cof}(\beta) & \left(a_0 \xrightarrow{\beta} a_1\right) \arrow[l] \arrow[ul] \arrow[dl] \\ & & a_0[1] & \end{tikzcd}$$

The datum of the above coCartesian sections boils down to two objects $a_0, a_1 \in \mathcal{D}(k)$ and two morphisms $\alpha, \beta : a_0 \to a_1$ in $\mathcal{D}(k)$. We might thus expect that $\mathcal{H}(\Gamma, \mathcal{F})$ is equivalent to the derived ∞-category of representations of the Kronecker quiver $Q = \; 1 \rightrightarrows 2 \;$. In the following, we sketch how this can be proven, by finding a compact generator of $\mathcal{H}(\Gamma, \mathcal{F})$.

Consider the following two coCartesian sections X_1, X_2:

$$\begin{tikzcd} & & 0 & \\ k & (k[-1] \to 0) \arrow[l] \arrow[ur] \arrow[dr] & k & (k[-1] \to 0) \arrow[l] \arrow[ul] \arrow[dl] \\ & & k & \end{tikzcd} \qquad \begin{tikzcd} & & k & \\ k & (0 \to k) \arrow[l] \arrow[ur] \arrow[dr] & k & (0 \to k) \arrow[l] \arrow[ul] \arrow[dl] \\ & & 0 & \end{tikzcd} \tag{13}$$

We would like to describe $\mathrm{Map}_{\mathcal{H}(\Gamma,\mathcal{F})}(X_a, X_b[i])$ for all $1 \leq a, b \leq 2$ and $i \in \mathbb{Z}$. For that, we first determine the morphisms between the local values of the sections. We have :

$$
\begin{aligned}
\mathrm{Map}_{\mathcal{D}(k)}(k, k[i]) &\simeq \begin{cases} k & i = 0 \\ 0 & \text{else}\,. \end{cases} \\
\mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{D}(k))}(k[-1] \to 0, 0 \to k[i]) &\simeq \begin{cases} k & i = 0 \\ 0 & \text{else}\,. \end{cases} \\
\mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{D}(k))}(0 \to k, k[-1+i] \to 0) &\simeq 0\,.
\end{aligned}
$$

The second equivalence can be seen as follows. Morphisms from $k[-1] \to 0$ to $0 \to k[i]$ correspond to commutative squares

$$
\begin{tikzcd}
k[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & k[i]
\end{tikzcd}
$$

in $\mathcal{D}(k)$. Since the pushout of the upper part is k, the datum of such a diagram is just equivalent to the datum of a morphism $k \simeq (k[-1])[1] \to k[i]$.

We now find $k \oplus k \subset \mathrm{Map}_{\mathcal{H}(\Gamma,\mathcal{F})}(X_1, X_2)$, where the first (second) direct summand consists of morphisms $X_1 \to X_2$ restricting at the vertex v_1 (v_2) of Γ to some non-zero morphism and at the vertex v_2 (v_1) to zero. Showing that these are all morphisms is more tricky, we will develop the necessary tools in some later lectures.

One can further show that there are no non-zero morphisms $X_1 \to X_2[i]$ for $i \neq 0$ nor $X_2 \to X_1[i]$ for i arbitrary. Finally, we have $\mathrm{Map}_{\mathcal{H}(\Gamma,\mathcal{F})}(X_1, X_1[i]) \simeq \mathrm{Map}_{\mathcal{H}(\Gamma,\mathcal{F})}(X_2, X_2[i]) \simeq \begin{cases} k & i = 0 \\ 0 & \text{else} \end{cases}$.

Definition 5.7. An object X in a stable and presentable ∞-category $\mathcal{C}$ is called a compact generator if

- X is a compact object, i.e. the functor $\mathrm{Map}_{\mathcal{C}}(X, \text{-}) : \mathcal{C} \to \mathcal{S}$ preserves filtered colimits.
- an object $Y \in \mathcal{C}$ is zero if and only if $\mathrm{Map}_{\mathcal{C}}(X, Y[i]) \in \mathcal{S}$ is contractible for all $i \in \mathbb{Z}$.

Theorem 5.8 ([Lur17, 7.1.7.2])**.** *Let $\mathcal{C}$ be a stable and presentable ∞-category and $X \in \mathcal{C}$ a compact generator. Then there exists an equivalence of ∞-categories $\mathcal{C} \simeq \mathrm{RMod}_{\mathrm{End}(X)}$, where $\mathrm{End}(X)$ is the endomorphism ring spectrum of X and $\mathrm{RMod}_{\mathrm{End}(X)}$ is the ∞-category of right $\mathrm{End}(X)$-module spectra.*

Lemma 5.9. *The object $X_1 \oplus X_2$ compactly generates $\mathcal{H}(\Gamma, \mathcal{F})$.*

The lemma will also follow from technology developed later on.

By our above computations, it follows that the endomorphism ring spectrum of $X_1 \oplus X_2 \in \mathcal{H}(\Gamma, \mathcal{F})$ is given by the path algebra of the Kronecker quiver and thus a discrete ring spectrum i.e. a ring. We have thus found the desired equivalence of ∞-categories, see also [Lur17, 7.1.2.13],

$$\mathcal{H}(\Gamma, \mathcal{F}) \simeq \mathrm{RMod}_{kQ} \simeq \mathcal{D}(kQ)\,.$$

The ∞-category $\mathcal{D}(kQ)$ is rather famous, it is also equivalent to the derived ∞-category of coherent sheaves on the complex projective line $\mathbb{P}^1$. The compact generators X_1 and X_2 correspond to the line bundles O and $O(1)$ and form an exceptional collection, which was observed by

Beilinson. There is a $\mathbb{P}^1$-family of skyscraper sheaves O_x, $x \in \mathbb{P}^1$. For $x \neq 0, \infty$, the skyscraper sheaf O_x corresponds to the following global section of $\mathcal{F}$:

$$\begin{tikzcd}
 & & k & & \\
0 & \left(k \xrightarrow{\cdot x} k\right) \arrow[l] \arrow[ur] \arrow[dr] & 0 & \left(k \xrightarrow{\mathrm{id}} k\right) \arrow[l] \arrow[ul] \arrow[dl] \\
 & & k[1] & &
\end{tikzcd}$$

We invite the reader to figure out the global sections corresponding to O_0, O_∞ and the line bundles $O(n)$ for $n \in \mathbb{Z}$.

Topological Fukaya categories

Topological Fukaya categories of marked surfaces were introduced by Dyckerhoff-Kapranov in [DK18] (in the $\mathbb{Z}/2\mathbb{Z}$-graded case) and in [DK15] (in the $\mathbb{Z}$-graded case). These categories may be seen as defined as the homotopy limits of constructible sheaves (or dual homotopy colimits of cosheaves) of enhanced triangulated categories on ribbon graphs. These sheaves were the first examples of parametrized perverse schobers and have no singularities. The definition of a parametrized perverse schober is thus simply a generalization of the approaches of [DK18,DK15] to allow singularities of the perverse schober, encoded via spherical adjunctions.

More generally, we will will use the following terminology.

Definition 5.10. Let $\mathbf{S}$ be a marked surface and Γ a spanning graph of $\mathbf{S}$. Given a Γ-parametrized perverse schober $\mathcal{F}$ on a marked surface without singularities, with generic stalk $\mathcal{N} \in \mathrm{St}$, we call its ∞-category of global sections a topological Fukaya category of $\mathbf{S}$ with values in $\mathcal{N}$.

Remark 5.11. In most treatments of topological Fukaya categories via sheaves of categories in the literature, e.g. [HKK17], they actually arise as the (homotopy) colimits of constructible cosheaves, meaning functors $\mathrm{Exit}(\Gamma)^{\mathrm{op}} \to \mathrm{St}$. Since we are free to work with presentable ∞-categories, by passing to Ind-completions if necessary, we can arrange these cosheaves to factor through $\mathcal{P}r^L_{\mathrm{St}} \to \mathrm{St}$. These cosheaves are simply the left adjoint diagram of a parametrized perverse schober and the colimit in $\mathcal{P}r^L_{\mathrm{St}}$ of the cosheaf is equivalent to the limit in $\mathcal{P}r^R_{\mathrm{St}}$ by Corollary 5.4.

In [HKK17] it was observed that topological Fukaya categories with values in $\mathcal{D}(k)$ are equivalent to the derived ∞-categories of graded gentle algebras. Their approach to the proof is very similar to the above example of the topological Fukaya category of the annulus. They identify some minimal set of objects which compactly generate the Fukaya category. Then, they check that their endomorphism dg-algebra is a gentle algebra.

The converse, that the derived category of any (homologically smooth) graded gentle algebra arises as a topological Fukaya category was proven in [LP20].

There is much more to say about these topological Fukaya categories, for example the more geometric perspective in terms of so called partially wrapped Fukaya categories (with objects=Lagrangians=curves in the surface,...). We will return to the problem of a geometric construction of objects (and Homs) in topological Fukaya categories (and more generally global sections of perverse schobers) in Lecture 7.

Contractions of ribbon graphs

The main result proven about topological Fukaya categories in [DK18, DK15] is that they do not depend on the choice of ribbon graph but only on the marked surface (and some extra data, such as a framing of the surface), up to equivalence. This is done by showing that the topological Fukaya category does not change under contractions of the ribbon graph and noting that any two ribbon graphs can be connected via a zig-zag of contractions. This is then used to show that the mapping class group of the marked surface acts on these categories. The independence on the choice of ribbon graph generalizes to perverse schobers, see below. Constructing mapping class group actions on global sections of perverse schobers remain an interesting open problem.

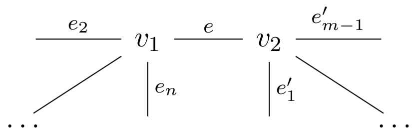
Let Γ be a ribbon graph and e an edge of Γ which is not a loop. Then there is a ribbon graph Γ', obtained from Γ by contracting the edge e and identifying the two adjacent vertices. We call Γ' the contraction of Γ at e.

Definition 5.12. We say that there is a contraction of ribbon graphs $c : \Gamma \to \Gamma'$, if Γ' is obtained from Γ by repeated[4] edge contractions from Γ.

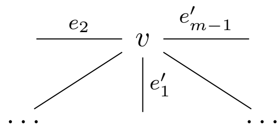
Proposition 5.13 ([Chr22a, Prop. 4.28])**.** *Let $\mathcal{F}$ be a Γ-parametrized perverse schober. Consider a contraction $c : \Gamma \to \Gamma'$, satisfying that no edges of Γ are contracted which connect two singularities of $\mathcal{F}$. Then there exists a Γ'-parametrized perverse schober $c_*\mathcal{F}$ and an equivalence of ∞-categories*

$$\mathcal{H}(\Gamma, \mathcal{F}) \simeq \mathcal{H}(\Gamma', c_*\mathcal{F})\,.$$

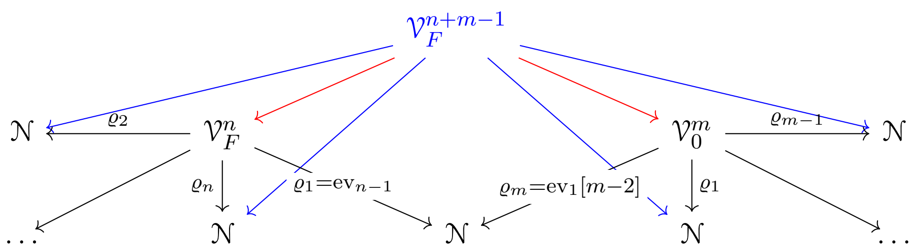
Proof sketch. We only deal with the local case that Γ consists of two vertices v_1, v_2 of valencies n, m and connected by the edge e:



The ribbon graph Γ' is given by a single vertex v and $n + m - 1$ edges:



Suppose now that v_2 is not a singularity of $\mathcal{F}$, meaning that at v_2, $\mathcal{F}$ is classified by the spherical functor $0 : 0 \to \mathcal{N}$. At v_1, $\mathcal{F}$ is classified by some spherical functor $F : \mathcal{V} \to \mathcal{N}$. Up to equivalence of perverse schobers, $\mathcal{F}$ is the non-colored part of the following diagram.



One can show that the square in the above diagram which contains the red arrows forms a pullback square (we invite the reader to check this on the level of objects, it is pretty obvious that this should be true), and the blue part of the diagram defines the perverse schober $c_*\mathcal{F}$. The assertion that $\lim \mathcal{F} \simeq \lim c_*\mathcal{F}$ now follows from the construction of $c_*\mathcal{F}$ locally as a limit (a precise argument can be made using that right Kan extensions commute with each other). $\square$

[4] We assume that all ribbon graphs have finitely many vertices and edges. In particular, a contraction can only contract finitely many edges.

Lecture 6: Relative Ginzburg algebras

Quivers with potential

Definition 6.1. A quiver Q consists of finite sets Q_0 of vertices and Q_1 of arrows, together with maps $s, t : Q_1 \to Q_0$ (source and target). A graded quiver is a quiver equipped with an integer labeling of each arrow.

Definition 6.2. Let Q be a quiver. The path algebra[5] kQ of Q is the k-algebra with

- underlying vector space k^P, where P is the set of paths in Q, meaning finite collections of formal composites of arrows of Q. The set P includes the lazy paths at the vertices of Q.
- composition of two paths $p_1 = a_n \dots a_1$, $p_2 = b_m \dots b_1$ defined by

$$p_2 p_1 = \begin{cases} b_m \dots b_1 a_n \dots a_1 & s(b_1) = t(a_n)\,, \\ 0 & \text{else.} \end{cases}$$

The lazy path $e_i \in kQ$ with $i \in Q_0$ satisfies $e_i^2 = e_i$ and we have $1_{kQ} = \sum_{j \in Q_0} e_j$.

If Q is graded, then the path algebra canonically inherits a grading.

Definition 6.3. Let Q be a quiver.

- The subset of cycles $kQ^{\mathrm{cyc}} \subset kQ$ is defined as the subspace spanned by cycles, i.e. paths $p = a_n \dots a_1$ satisfying that $s(a_1) = t(a_n)$. Lazy paths are not allowed in kQ^{cyc}.
- A potential W of Q is an element $W \in kQ^{\mathrm{cyc}}$.
- Let $a \in Q_1$. The cyclic derivative by a is the linear map $\partial_a : kQ^{\mathrm{cyc}} \to kQ$ defined on a cycles c by
$$\partial_a(c) = \sum_{c = uav} vu$$
where v, u are some, possibly lazy, paths in Q.

Definition 6.4. An ice quiver (Q, F) consists of a quiver Q and a subquiver F of Q called the frozen subquiver.

Relative Ginzburg algebras

Ginzburg algebras are a class of dg-algebras, first introduced in [Gin06]. Their derived categories have been used for, among other things, the categorification of cluster algebras, see [Kel12] for a survey, and the algebraic description of Fukaya categories, see for example [Smi15]. Ginzburg algebras are constructed from a quiver with potential. A recent generalization of Ginzburg algebras are so called relative Ginzburg algebras. They were introduced for ice quivers in general in [Wu23] and for ice quivers arising from triangulated surfaces independently in [Chr22a].

Definition 6.5. Let (Q, F, W) be an ice quiver with potential. Let $\tilde{Q}$ be the graded quiver with vertices $\tilde{Q}_0 = Q_0$ and

- an arrow $a : i \to j$ in degree 0 for every such arrow of Q,
- an arrow $a^* : j \to i$ in degree 1 for every non-frozen arrow $a : i \to j$ of Q.
- a loop $l_i : i \to i$ in degree 2 for every non-frozen vertex $i \in Q_0 \backslash F_0$.

[5] We do not consider any completions of path algebras.

The relative Ginzburg algebra $\mathscr{G}_{(Q,F,W)}$ is defined as the dg-algebra whose underlying graded algebra is the graded path algebra of $\tilde{Q}$ and whose differential d is defined on the generators (arrows) as follows.

- $d(a) = 0$ for all arrows $a \in Q_1$.
- $d(a^*) = \partial_a W$ for all non-frozen arrows $a \in Q_1 \backslash F_1$.
- $d(l_i) = \sum\limits_{a \in Q_1 \backslash F_1} e_i[a, a^*]e_i$ for all $i \in Q_0 \backslash F_0$ (note that since i is unfrozen, all arrows starting or ending at i are also unfrozen).

Relative higher Ginzburg algebras from n-angulated surfaces

Definition 6.6. Let $n \geq 3$. An ideal n-angulation of a marked surface $\mathbf{S}$ is a spanning graph $\mathcal{T}$ of $\mathbf{S}$, such that every vertex of $\mathcal{T}$ has valency n. Ideal 3-angulations are also called ideal triangulations.

We call the edges of $\mathcal{T}$ with one end in $\partial\mathbf{S}$ the external edges and the other edges internal.

Remark 6.7. The datum of an ideal triangulation $\mathcal{T}$ of a marked surface is equivalent, via passing to the 'dual', to the datum of a decomposition of the surface into triangles (this is usually called an ideal triangulation). The 'dual' has a triangle for each vertex of $\mathcal{T}$ and two triangles share an edge if and only if the corresponding vertices of $\mathcal{T}$ are connected by an edge of $\mathcal{T}$. If $\mathcal{T}$ has loops, then some of the triangles are self-folded, meaning that two of their edges coincide.

Construction 6.8. Let $\mathbf{S}$ be a marked surface with an ideal n-angulation $\mathcal{T}$. Assume for simplicity, that $\mathcal{T}$ has no loops. We define the ice quiver $(Q_\mathcal{T}, F_\mathcal{T})$ as follows:

- The vertices of $Q_\mathcal{T}$ are the edges of $\mathcal{T}$ (both the internal and external).
- For each vertex v of $\mathcal{T}$ and ordered pair of incident edges $i \neq j$, there are is an arrow $a_{v,i,j} : i \to j$. The degree of $a_{v,i,j}$ is $l - 1$, where l is the number of positions in the counterclockwise cyclic order at v by which i follows j.
- The frozen subquiver F consists of the vertices of $Q_\mathcal{T}$ arising from external edges.

We denote by $Q_\mathcal{T}^\circ$ the full subquiver of $Q_\mathcal{T}$ spanned by the non-frozen vertices.

Definition 6.9. Let $\mathcal{T}$ be an ideal n-angulation of a marked surface $\mathbf{S}$. Let $\tilde{Q}_\mathcal{T}$ be the graded quiver with the same vertices and arrows of $Q_\mathcal{T}$ and additionally loops $l_i : i \to i$ of degree $n-1$ at the non-frozen vertices. The relative (higher) Ginzburg algebra $\mathscr{G}_\mathcal{T}$ of $\mathcal{T}$ is the dg-algebra whose underlying graded algebra is given by the graded path algebra $\tilde{Q}_\mathcal{T}$ and with the differential determined on generators by

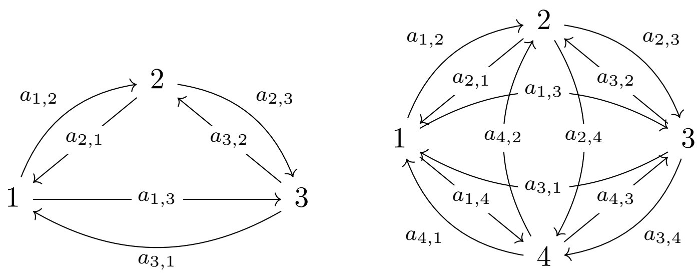
- $d(a_{v,i,j}) = \sum_{j<k<i} (-1)^{|a_{v,k,j}|} a_{v,k,j} a_{v,i,k}$, where the sum runs over all k appearing between j and i in the cyclic order.
- $d(l_i) = \sum_{v,\, j \neq i} (-1)^{|a_{v,i,j}|+1} a_{v,j,i} a_{v,i,j}$, where the sum runs over all $v \in \mathcal{T}_0$ incident to i.

By starting with with the full subquiver of $Q_\mathcal{T}$ consisting of non-frozen vertices, one obtains an analogously defined dg-algebra $\mathscr{G}_\mathcal{T}^\circ$, called the non-relative (higher) Ginzburg algebra of $\mathcal{T}$.

Remark 6.10. Consider the case $n = 3$. The graded quiver $Q_{\mathcal{T}}$ is the double (meaning one adds dual arrows in degree 1) of an ungraded ice quiver consisting of clockwise 3-cycles at the vertices. This quiver admits a potential $W'_{\mathcal{T}}$ consisting of these clockwise 3-cycles. Then $\mathscr{G}_{\mathcal{T}}$ is the relative Ginzburg algebra of this ice quiver with potential. Similarly, $\mathscr{G}^{\circ}_{\mathcal{T}}$ is analogously the non-relative Ginzburg algebra of the non-frozen part of the quiver.

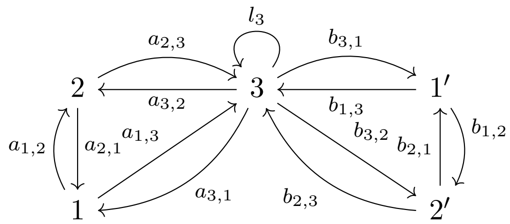
For $n > 3$, we expect the dg-algebra $\mathscr{G}_{\mathcal{T}}$ can be embedded in a more general definition of relative higher Ginzburg algebra associated to a suitable graded quiver with a graded potential. The main issue with making this precise is getting all the signs right. Note however that being a Ginzburg algebra is not a property (like being a gentle algebra) and there might be no unique, reasonable and most general construction of relative higher Ginzburg algebras.

Example 6.11. Near the vertices of $\mathcal{T}$ of valency $n = 3$ or $n = 4$, the quiver $Q_{\mathcal{T}}$ looks as follows.



The relative Ginzburg algebras of the 3-gon and 4-gon are given by the the path algebras of the above quivers with differentials. In the $n = 3$ case, the differentials are $d(a_{2,1}) = a_{3,1}a_{2,3}$, $d(a_{3,2}) = a_{2,3}a_{1,2}$, $d(a_{3,2}) = a_{3,1}a_{1,2}$ and $d(a_{1,2}) = d(a_{2,3}) = d(a_{3,1}) = 0$. In the $n = 4$ case, we have the differentials $d(a_{1,2}) = 0$, $d(a_{1,3}) = a_{2,3}a_{1,2}$ and $d(a_{1,4}) = a_{2,4}a_{1,2} - a_{3,4}a_{1,3}$. $d(a_{2,1}) = a_{4,1}a_{2,4} - a_{3,1}a_{2,3}$, $d(a_{3,1}) = a_{4,1}a_{3,4}$, $d(a_{4,1}) = 0$ and so on.

Example 6.12. Let $n = 3$ and $\mathbf{S}$ be the unpunctured 4-gon and $\mathcal{T}$ an ideal 3-angulation consisting of two vertices. The relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$ is the path-algebra of the graded quiver:



The differential of the arrows in degree 0 vanish, and in degree 1 are as in Example 6.11. The differential of l_3 is $d(l_3) = a_{2,3}a_{3,2} + b_{2,3}b_{3,2} - a_{1,3}a_{3,1} - b_{1,3}b_{3,1}$.

Relative Ginzburg algebras via perverse schobers

Recall from Exercise 4, the spherical adjunction $\psi_! : \mathcal{D}(k[t_{n-2}]) \to \mathcal{D}(k) : \psi^*$, where $k[t_{n-2}]$ is the graded polynomial algebra with generator t_{n-2} in degree $n-2$ and $\psi : k[t_{n-2}] \xrightarrow{t_{n-2}\mapsto 0} k$.

Proposition 6.13. *Let $\mathbf{S}$ be the unpunctured n-gon and $\mathcal{T}$ the unique ideal n-angulation. There exists an equivalence of ∞-categories*

$$\mathcal{D}(\mathscr{G}_{\mathcal{T}}) \simeq \mathcal{V}^n_{\psi^*}$$

Lemma 6.14. *Let $n \geq 1$ and $F : \mathcal{V} \leftrightarrow \mathcal{N} : G$ be a spherical adjunction of stable and presentable ∞-categories. Assume that $F(a) \simeq 0$ if and only if $a \simeq 0$ for all $a \in \mathcal{V}$ (this means that F is conservative). If $b \in \mathcal{N}$ is a compact generator, then $\bigoplus_{i=1}^n \varsigma_i(b) \in \mathcal{V}^n_F$ is a compact generator.*

Proof. The functors ς_i preserve all compact objects because they admit colimit preserving right adjoints. It follows that $\bigoplus_{i=1}^n \varsigma_i(b) \in \mathcal{V}^n_F$ is compact. Let $X = (a \to b_1 \to \cdots \to b_{n-1}) \in \mathcal{V}^n_F$ be an object with

$$\mathrm{Map}_{\mathcal{V}^n_F}(\bigoplus_{i=1}^n \varsigma_i(b), X[l]) \simeq * \text{ (contractible)}$$

for all $l \in \mathbb{Z}$. Using the adjunctions $\varsigma_i \dashv \varrho_{i-1}$ and $\varsigma_1 \dashv T_{\mathcal{N}}[n-1]\varrho_n$, we obtain that

$$\mathrm{Map}_{\mathcal{N}}(b, \varrho_i(X)[l]) \simeq *$$

for all $1 \leq i \leq n-1$ and

$$\mathrm{Map}_{\mathcal{N}}(b, T_{\mathcal{N}}\varrho_n(X)[l+n-1]) \simeq * \,.$$

Since b is a compact generator and $T_{\mathcal{N}}$ and equivalence, it follows that $\varrho_i(X) \simeq 0$ for all $1 \leq i \leq n$. We thus have $b_{n-1} \simeq \varrho_1(X) \simeq 0$. Similarly, we find $b_{n-2}[1] \simeq \varrho_2(X) \simeq 0$ and so on. It follows that $b_1, \ldots b_{n-1} \simeq 0$. Finally, we have $F(a)[n] \simeq \varrho_1(X) \simeq 0$, which implies $a \simeq 0$ by assumption. This shows that $X \simeq 0$. We conclude that $\bigoplus_{i=1}^n \varsigma_i(b) \in \mathcal{V}^n_F$ is a compact generator. □

Proof sketch of Proposition 6.13. Generalizing Example 4.7, there exists an equivalence between $\mathcal{V}^n_{\psi^*}$ and the derived ∞-category of the upper triangular dg-algebra

$$A = \begin{pmatrix} k & k & 0 & \ldots & 0 & 0 \\ 0 & k[t_{n-2}] & k[t_{n-2}] & 0 & \ldots & 0 \\ 0 & 0 & k[t_{n-2}] & k[t_{n-2}] & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & k[t_{n-2}] & k[t_{n-2}] \\ 0 & 0 & 0 & \ldots & 0 & k[t_{n-2}] \end{pmatrix}$$

We have the following correspondence between A-modules and objects in $\mathcal{V}^n_{\psi^*}$ with $2 \leq i \leq n$.

$$k \sim (k[-n+1] \to 0 \to \cdots \to 0)$$

$$n-i+2\text{-th diagonal } k[t_{n-2}] \sim \left(0 \to \cdots \to \underbrace{k[t_{n-2}][-i+2]}_{\text{vertex } n-i+1} \to \cdots \to 0 \right) = \varsigma_i(k[t_{n-2}])$$

The object $\varsigma_1(k[t_{n-2}]) = (k \to k[t_{n-2}] \to \cdots \to k[t_{n-2}])$ corresponds to some cofibrant A-module

$$X = \mathrm{cone}(\ldots \mathrm{cone}(k \to k[t_{n-2}]) \cdots \to k[t_{n-2}]) \in \mathcal{D}(A)\,.$$

The object $\bigoplus_{i=1}^n \varsigma_i(k[t_{n-2}])$ is a compact generator of $\mathcal{V}^n_{\psi^*}$, so that the corresponding A-module M is a compact generator of $\mathcal{D}(A)$. We thus have $\mathcal{D}(A) \simeq \mathcal{D}(\mathrm{End}(M))$, see Theorem 5.8. We can explicitly determine $\mathrm{End}(M)$ as a dg-algebra by using that $\mathrm{End}(M) \simeq \mathrm{RHom}_A(M, M)$, and since M is cofibrant, $\mathrm{RHom}(M, M) = \mathrm{Hom}(M, M)$ is the dg-algebra of dg-endomorphism in the dg-category dgMod_A. We have now reduced the problem to a pure computation of a dg-endomorphism algebra. This is some work, but can be done, yielding an explicit quasi-isomorphism

$$\mathrm{End}(M) \simeq \mathscr{G}_{\mathcal{T}}$$

of dg-algebras. □

Remark 6.15. Under the equivalence $\mathcal{D}(\mathscr{G}_{\mathcal{T}}) \simeq \mathcal{V}^n_{\psi^*}$, the object $\varsigma_i(k[t_{n=2}])$ is mapped to the projective module $P_i = e_i \mathscr{G}_{\mathcal{T}}$ of the vertex i of the quiver $Q_{\mathcal{T}}$. Note also the similarity of the rotational symmetry of $\mathscr{G}_{\mathcal{T}}$ and the rotational symmetry $T_{\mathcal{V}^n_{\psi^*}}$ of $\mathcal{V}^n_{\psi^*}$.

Exercise 5. Let $\mathcal{T}$ be the unique ideal n-angulation of the n-gon. Show that

$$\mathrm{RHom}_{\mathscr{G}_{\mathcal{T}}}(e_i\mathscr{G}_{\mathcal{T}}, e_j\mathscr{G}_{\mathcal{T}}) \simeq \begin{cases} k[t_{n-2}] & \text{if } j = i, i+1 \\ 0 & \text{else} \end{cases}$$

using Remark 6.15 by computing the Ext-groups between the ς_i's. Try to describe the corresponding homology classes of $\mathscr{G}_{\mathcal{T}}$.

Theorem 6.16 ([Chr22a, Chr21])**.** *Let $\mathcal{T}$ be an ideal triangulation of a marked surface. There exists a $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_{\mathcal{T}}$, which is at every vertex of $\mathcal{T}$ classified by the spherical adjunction $\psi^* \dashv \mathrm{radj}(\psi^*)$, and an equivalence of ∞-categories.*

$$\mathcal{D}(\mathscr{G}_{\mathcal{T}}) \simeq \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})\,.$$

Under this equivalence, the derived ∞-category $\mathcal{D}(\mathscr{G}_{\mathcal{T}}^{\circ})$ of the non-relative Ginzburg algebra arises as the full subcategory of $\mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$ consisting of global sections which vanish on all external edges of $\mathcal{T}$.

Proof idea. The perverse schober $\mathcal{F}_{\mathcal{T}}$ is glued from copies of $\mathcal{F}_n(\psi^*)$, one for each vertex of $\mathcal{T}$. We discuss the choice of $\mathcal{F}_{\mathcal{T}}$ in Remark 6.17 below. To compute its limit, we use Corollary 5.4, and instead compute the colimit over the left adjoint diagram. The functor $\mathcal{D}(\text{-}) : \mathrm{dgCat}_k \to \mathcal{P}r^L$ maps homotopy colimits with respect to the quasi-equivalence model structure to colimits in $\mathcal{P}r^L$. The right adjoint diagram of $\mathcal{F}_{\mathcal{T}}$ lies in the image of $\mathcal{D}(\text{-})$ and and the computation of its colimit thus boils down to computing the homotopy colimit of a diagram $\mathrm{Exit}(\mathcal{T})^{\mathrm{op}} \to \mathrm{dgCat}_k$, which assigns

- to the vertices of $\mathcal{T}$ a multi-object version of the relative Ginzburg algebra of the n-gon.
- to the edges the graded polynomial algebra $k[t_{n-2}]$, which can be seen as the $(n-1)$-CY Ginzburg algebra of the A_1-quiver.

Computing this homotopy colimit is rather manageable, it is found to match the relative Ginzburg algebra $\mathscr{G}_{\mathcal{T}}$. □

Two examples, where the computations for the proof of Theorem 6.16 are illustrated, can be found in [Chr22a, Section 6.2].

Remark 6.17. While locally, any perverse schober is determined up to equivalence by the spherical adjunctions, this is not at all true globally. There are different equivalent descriptions of the $\mathcal{F}_n(\psi^*)$, which are no longer equivalent after gluing. We are free to restrict ourselves to gluing those diagram arising from a choice of total orders of the edges at the vertices of $\mathcal{T}$. The twist functor T of the adjunction $\psi^* \dashv \mathrm{radj}(\psi^*)$ is equivalent to $\phi^*[1-n]$, with $\phi : k[t_{n-2}] \xrightarrow{t_{n-2} \mapsto (-1)^n t_{n-2}} k[t_{n-2}]$. Thus if n is even, $T[n-1] \simeq \mathrm{id}_{\mathcal{D}(k[t_{n-2}])}$ and if n is odd $(T[n-1])^2 \simeq \mathrm{id}_{\mathcal{D}(k[t_{n-2}])}$. By Proposition 4.16, this means that the different choices of total order do not change $\mathcal{F}_n(\psi^*)$ much (if n is odd) or at all (if n is even)! Up to some sign issues in the case that n is odd, there is thus an essentially unique way to define $\mathcal{F}_{\mathcal{T}}$. This nice feature is the main reason we restrict to n-angulated surfaces, instead of arbitrary polygonal surfaces.

Outlook: Further classes of examples of perverse schobers

Recall from Remark 3.9, that every spherical fibration $X \to Y$, e.g. $S^n \to *$, gives us a spherical adjunction. Many examples of spherical adjunctions arising in algebra also arise this way. For example, the spherical adjunction $\psi_! : \mathcal{D}(k[t_{n-2}]) \leftrightarrow \mathcal{D}(k) : \psi^*$ studied in this lecture is equivalent to the adjunction $f_! : \mathrm{Fun}(S^{n-1}, \mathcal{D}(k)) \leftrightarrow \mathcal{D}(k) : f^*$ arising from $f : S^{n-1} \to *$,

spherical adjunction(s)	spherical fibration	global sections
$0 \leftrightarrow \mathcal{D}(k)$	/	graded gentle algebra = topological Fukaya category
$0 \leftrightarrow \mathcal{D}(k)$, $\mathcal{D}(k[\epsilon]/\epsilon^2 = \epsilon) \leftrightarrow \mathcal{D}(k)$	$S^0 \to *$	graded skew-gentle algebra
$\mathcal{D}(k) \leftrightarrow \mathcal{D}(k[t_0^{\pm}])$ with $\lvert t_0\rvert = 0$	$S^1 \to *$	multiplicative preprojective algebra?
$\mathcal{D}(k) \leftrightarrow \mathcal{D}(k[t_{n-2}])$, $\lvert t_{n-2}\rvert = n-2$	$S^{n-1} \to *$, $n-1 \geq 2$	relative n-CY Ginzburg algebra
'fixed points' of $\mathcal{D}(k) \leftrightarrow \mathcal{D}(k[t_{n-2}])$	/	relative graded Brauer graph algebras
$0 \leftrightarrow \mathcal{D}(k[t_{n-2}^{\pm}])$	/	generalized $(n-1)$-Calabi–Yau cluster category of unpunctured marked surface = $(n-1)$-periodic topological Fukaya category
$0 \leftrightarrow \mathcal{D}(k[t_1^{\pm}])$, $\mathcal{D}(k[t_2^{\pm}]) \leftrightarrow \mathcal{D}(k[t_1^{\pm}])$	/	generalized 2-Calabi–Yau cluster category of punctured marked surface
$\mathcal{D}(k) \leftrightarrow \mathcal{D}(k[t_1])$, $\mathcal{D}(k[t_2]) \leftrightarrow \mathcal{D}(k[t_1])$	$S^2 \to *$, $S^3 \xrightarrow{Hopf} S^2$	3-CY 'lift' of above generalized cluster category

Table 3: Some more classes of examples of perverse schobers in algebra, arising from spherical adjunction arising from spherical fibrations and the corresponding ∞-categories of global sections.

see [Chr22a, Prop. 5.5]. A few such examples, and the global sections of perverse schobers locally classified by these adjunctions, are collected in Table 3. There are surely many more interesting examples of this kind.

The appearance of spherical adjunctions arising from spherical fibrations can be motivated by the (expected) relation between perverse schobers and Fukaya categories of Lefschetz fibrations (with typical fiber, for example, a sphere). Giving more details on the relation between such Fukaya categories and perverse schober would lead us too far astray from the plan of the course. The interested reader may find some more explanations in [KS14, Chr22a, Chr21] and specifically for Fukaya-Seidel categories also in [KSS20].

Lecture 7: Geometric models I

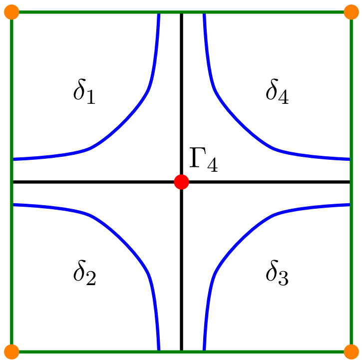
In this lecture, we describe some of the global sections of a perverse schober in terms of certain curves in the underlying surface. The main result is that a compact generator always exists (under mild assumptions) and that it can be described by such objects arising from curves.

Global sections of perverse schobers via curves: locally

Let $F : \mathcal{V} \to \mathcal{N}$ be a spherical functor and $n \geq 2$. Consider the once-punctured n-gon $\mathbf{D}_n$ with its spanning graph Γ_n, see Figure 2. The spherical functor F defines us a perverse schober on $\mathbf{D}_n$ with global sections $\mathcal{V}^n_F$. Given $L \in \mathcal{N}$, we have the objects $\varsigma_i(L) \in \mathcal{V}^n_F$ with $1 \leq i \leq n$. By the adjunction $\varrho_j \dashv \varsigma_j$, we find

$$\operatorname{Map}(\varsigma_i(L), \varsigma_j(L)) \simeq \operatorname{Map}(\varrho_j\varsigma_i(L), L) \simeq \begin{cases} \operatorname{Map}(L, L) & i = j \text{ or } i = j+1 \neq 1 \\ \operatorname{Map}(T_{\mathcal{N}}(L)[n-1], L) & j = n, i = 1 \\ 0 & \text{else.} \end{cases}$$

We think of $\varsigma_i(L)$ as an object arising from a curve δ_i in $\mathbf{D}_n \backslash M$, which starts at a boundary component and then goes one step around the puncture to the next boundary component. For $n = 4$, this can be depicted as follows:



We can describe the morphism objects between the $\varsigma_i(L)$'s in terms of the directed boundary intersections of these curves.

Definition 7.1. Let $\mathbf{S}$ be a marked surface and γ, γ' curves in $\mathbf{S}\backslash M$ considered up to homotopy relative $\partial\mathbf{S}\backslash M$. Suppose that the representatives of their homotopy classes have no intersections or are chosen with the minimal number of intersections. A directed boundary intersection from γ to γ' is an intersection of both γ and γ' with a boundary component B of $\mathbf{S}\backslash M$, such that the intersection of γ with B precedes the intersection of γ' with B in the clockwise orientation of B.

There is a directed boundary intersection from $\delta_{i+1}(L)$ to $\delta_i(L)$. We denote $\operatorname{Ext}^i(X,Y) \coloneqq \operatorname{Map}(X, Y[-i])$ and $\operatorname{Ext}^*(X,Y) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^i(X,Y)$. We have:

Proposition 7.2. *Let $i \neq j$ and suppose that $T_{\mathcal{N}}(L)[n-1] \simeq L[m]$ for some $m \in \mathbb{Z}$. Then $\operatorname{Ext}^*(\varsigma_i(L), \varsigma_j(L))$ counts the number of directed boundary intersections from $\delta_i(L)$ to $\delta_j(L)$ in terms of $\operatorname{Ext}^*(L,L)$.*

Example 7.3.

1) For $\mathcal{V} = 0$ and $\mathcal{N} = \mathcal{D}(k)$, we have $\mathcal{V}^n_F \simeq \mathcal{D}(kA_{n-1})$ is the derived ∞-category of the A_{n-1}-quiver. In this case, $T_{\mathcal{N}} \simeq [-1]$ and thus $T_{\mathcal{N}}(k)[n-1] \simeq k[n-2]$. We recover some of the geometric model for the derived category of the A_n-quiver, which is a special case of the geometric model for gentle algebras [OPS18].

2) For $F = \psi^* : \mathcal{D}(k) \to \mathcal{D}(k[t_{n-2}])$, we have $\mathcal{V}^n_F \simeq \mathcal{D}(\mathscr{G}_{\mathcal{T}})$ for $\mathcal{T}$ the ideal n-angulation of the n-gon. We further have $T_{\mathcal{N}}(L)[n-1] \simeq L$ for all $L \in \mathcal{D}(k[t_{n-2}])$. In this case the $\varsigma_i(L)$ are the projective $\mathscr{G}_{\mathcal{T}}$-modules associated to the vertices of the quiver $Q_{\mathcal{T}}$.

For perverse schobers on the once-punctured n-gon, we can thus describe a compact generator (recall Lemma 6.14) of the global sections in terms of curves and the Homs in terms of intersections. The goal of this and the final lecture is to generalize this argument to the global sections of perverse schobers on arbitrary marked surfaces.

Local sections of perverse schobers and gluing of sections

Let $\mathcal{F}$ be a Γ-parametrized perverse schober with generic stalk $\mathcal{N}$. Consider the Grothendieck construction $p : \Gamma(\mathcal{F}) \to N(\operatorname{Exit}(\Gamma))$ of $\mathcal{F}$.

Definition 7.4. The ∞-category $\mathcal{L}(\Gamma, \mathcal{F})$ of local sections (or simply sections) of $\mathcal{F}$ is defined as the ∞-category $\operatorname{Fun}_{\operatorname{Exit}(\Gamma)}(\operatorname{Exit}(\Gamma), \Gamma(\mathcal{F}))$ of all sections of p.

The ∞-category $\mathcal{H}(\Gamma, \mathcal{F}) \subset \mathcal{L}(\Gamma, \mathcal{F})$ is the full subcategory of coCartesian sections. One should think of the ∞-category $\mathcal{L}(\Gamma, \mathcal{F})$ as a categorification of the direct sum of the stalks

$\bigoplus_{x\in\mathrm{Exit}(\mathcal{T})} \underline{H}_{\Gamma}(F)_x$ at $x = v, e$ the vertices and edges of Γ, for F a perverse sheaf. In complete analogy to sheaves of vector spaces, one can glue local sections which agree on the overlap. Gluing sufficiently many compatible local sections, one gets a global sections. The technology of perverse schobers thus allows us to give local-to-global constructions of objects in some interesting ∞-categories!

Let us describe how we can glue two local sections, each on a once-punctured n-gon. Let v be a vertex of Γ of valency $n \geq 2$, $1 \leq i \leq n$ and $L \in \mathcal{N}$. At v, $\mathcal{F}$ is classified by a spherical functor $F : \mathcal{V}_v \to \mathcal{N}$. To simplify notation, assume that $\mathcal{F}|_{\mathrm{Exit}(\Gamma)_{v/}} = \mathcal{F}_n(F)$, see Definition 4.12, and that the corresponding twist functor satisfies $T_{\mathcal{N}} \simeq [1-n]$. The object $\varsigma_i(L) \in \mathcal{V}^n_F$ gives rise to a local section M^L_{v,δ_i} of $\mathcal{F}$ which is near v given by

$$\begin{tikzcd}
\cdots & \varrho_{i+1}\varsigma_i(L)\simeq 0 & \\
\varrho_{i-2}\varsigma_i(L)\simeq 0 & \varsigma_i(L) \arrow[l] \arrow[r] \arrow[u] \arrow[d] \arrow[ul] & \varrho_i\varsigma_i(L)\simeq L \\
& \varrho_{i-1}\varsigma_i(L)\simeq L &
\end{tikzcd}$$

and vanishes on the remainder of $\mathrm{Exit}(\Gamma)$.

Let e be an edge of Γ. We denote by Z^L_e the local section of $\mathcal{F}$ given by

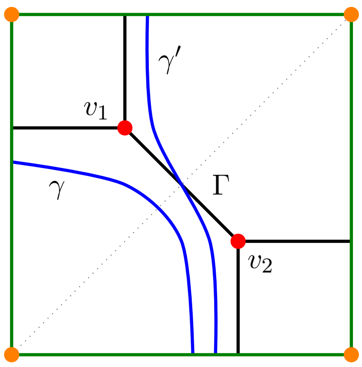
$$\begin{aligned} Z^L_e : \mathrm{Exit}(\mathcal{T}) &\longrightarrow \Gamma(\mathcal{F}) \\ e &\longmapsto L \\ e \neq x &\longmapsto 0 \end{aligned}$$

Let v_1, v_2 be two vertices of Γ incident to e. Then $e = e_i$ at v_1 and $e = e_j$ at v_2 in the chosen total orders. We have $M^L_{v_1,\delta_i}(e) \simeq M^L_{v_2,\delta_j}(e) \simeq Z^L_e(e) \simeq L$ and there thus exist morphisms $Z^L_e \to M^L_{v_1,\delta_i}(e), M^L_{v_2,\delta_j}(e)$ restricting to the identity on L at e and vanishing otherwise. The pushout of the diagram in $\mathcal{L}(\Gamma,\mathcal{F})$

$$\begin{tikzcd}
& Z^L_e \arrow[dl] \arrow[dr] & \\
M^L_{v_1,\delta_i} & & M^L_{v_2,\delta_j}
\end{tikzcd}$$

gives us a new section, which should be thought of as the gluing of $M^L_{v_1,\delta_i}(e)$ and $M^L_{v_2,\delta_j}(e)$ on their compatible overlap Z^L_e.

Example 7.5. We spell the above pushout out in an example. Consider the twice punctured 4-gon with the following spanning graph Γ:



Consider the following Γ-parametrized perverse schober:

$$\begin{tikzcd}
 & \mathcal{N} & & & \\
\mathcal{N} & \mathcal{V}^3_{F_1} \arrow[l, "\varrho_2"] \arrow[u, "\varrho_1"] \arrow[r, "\varrho_3"] & \mathcal{N} & \mathcal{V}^3_{F_2} \arrow[l, "\varrho_1"] \arrow[r, "\varrho_3"] \arrow[d, "\varrho_2"] & \mathcal{N} \\
 & & & \mathcal{N} &
\end{tikzcd}$$

The sections $M^L_{v_1,\delta_3}$ and $M^L_{v_2,\delta_2}$ can be glued along Z^L_e, with e the diagonal edge of Γ. The resulting global section, denoted M^L_γ, is given as follows[6].

$$\begin{tikzcd}
 & 0 & & & \\
L & (0 \to L[-1] \to 0) \arrow[l] \arrow[u] \arrow[r] & L & (0 \to 0 \to L) \arrow[l] \arrow[r] \arrow[d] & 0 \\
 & & & L &
\end{tikzcd}$$

Similarly, the sections $M^L_{v_1,\delta_1}$ and $M^L_{v_2,\delta_2}$ can also be glued along Z^L_e, yielding the following global section, denoted $M^L_{\gamma'}$.

$$\begin{tikzcd}
 & L & & & \\
0 & (G_1(L) \to L \to L) \arrow[l] \arrow[u] \arrow[r] & L & (0 \to 0 \to L) \arrow[l] \arrow[r] \arrow[d] & 0 \\
 & & & L &
\end{tikzcd}$$

Above, G_1 denotes the right adjoint of the spherical functor F_1. One should think of these gluings as follows: One starts by some two curves on the two 3-gons contained in the 4-gon. The condition that the corresponding local sections agree on the overlap is equivalent to the condition that the curves are composable. By composing these two curves, one obtains a curve in the 4-gon (γ or γ') with endpoints on the boundary of the surfaces, which has an associated global section.

Objects via curves: globally

Definition 7.6. Let $\mathbf{S}$ be a marked surface and Γ a spanning graph. Let v be a vertex of Γ of valency $n \geq 2$. A segment at v is a curve in a contractible neighborhood of v which starts at one edge e of Γ incident to v and ends at an adjacent (in the cyclic order) edge e' of Γ incident to v.

Definition 7.7. An open pure matching curve in $\mathbf{S}$ is a curve $\gamma : [0,1] \to \mathbf{S} \backslash M$, composed of finitely many segments, such that no segments lying at the same vertex are composed with each other.

Construction 7.8. Let $\mathcal{F}$ be a Γ-parametrized perverse schober with generic stalk $\mathcal{N}$. Let $L \in \mathcal{N}$. For every pure matching curve γ, one can construct a global section $M^L_\gamma \in \mathcal{H}(\Gamma, \mathcal{F})$ via gluing.

[6] This is easy to see, using that limit and colimits of sections are computed pointwise in $\mathrm{Exit}(\Gamma)$.

To simplify notation, we first do the case that at each vertex v of Γ of valency n, the cotwist functor of the spherical functor F_v satisfies $T_{\mathcal{N}}(L) \simeq L[1-n]$. For now, also assume that $\mathcal{F}|_{\mathrm{Exit}(\Gamma)_{/v}} = \mathcal{F}_n(F_v)$. We define $M^L_\gamma := M^L_{\gamma \leq N}$, which is a global section.

Let γ be an open matching curve in $\mathbf{S}$, composed of the segments δ_i, $i \in I = \{1, \dots, N\}$. Let $M^L_{\delta_i}$ be the local section corresponding to δ_i and denote by e_i the edge of Γ where δ_i ends. We define recursively the section $M^L_{\gamma \leq i}$ as the pushout of the following diagram in $\mathcal{L}(\Gamma, \mathcal{F})$:

$$\begin{tikzcd} & Z^L_{e_i} \arrow[dl, hook] \arrow[dr, hook] & \\ M^L_{\gamma \leq i-1} & & M^L_{\delta_i} \end{tikzcd}$$

The morphism in the above diagrams are the apparent inclusions of $Z^L_{e_i}$.

For $\mathcal{F}$ an arbitrary perverse schober, the above construction can be modified, by keeping track of some monodromy data, denoted μ_i. Let δ_i be a segment wrapping in the counterclockwise direction. Let v_i be the vertex of Γ where δ_i lies. Define $\mu_i : \mathcal{F}(v_i \to e_{i+1}) \circ \mathrm{ladj}(\mathcal{F}(v_i \to e_i) : \mathcal{N} \to \mathcal{N}$ and $\mu_{\leq i} = \mu_i \circ \dots \mu_1$. If δ_i goes clockwise, we replace ladj by radj in the above definition. If δ_i goes counterclockwise, we define $M^L_{\delta_i}$ as the following local section at v_i:

$$\begin{tikzcd} \dots & 0 & \\ 0 & \mathrm{ladj}(\mathcal{F}(v_i \to e_i)(\mu_{<i}(L)) \arrow[l] \arrow[r] \arrow[u] \arrow[ul] \arrow[d] & \mu_{<i+1}(L) \\ & \mu_{<i}(L) & \end{tikzcd}$$

If δ_i goes clockwise, define $M^L_{\delta_i}$ analogously and with ladj replaced by radj. These $M^L_{\delta_i}$ can again be glued to a global section M^L_γ.

Exercise 6. Show that the section M^L_γ from Construction 7.8 is a global section.

Compact generators

Let $\mathcal{F}$ be a Γ-parametrized perverse schober with generic stalk $\mathcal{N}$.

Definition 7.9. Let $e \in \mathrm{Exit}(\Gamma)$ be an edge of Γ. We denote by $\mathrm{ev}_e : \mathcal{H}(\Gamma, \mathcal{F}) \to \mathcal{F}(e)$ the functor which evaluates a coCartesian section $\mathrm{Exit}(\Gamma) \to \Gamma(\mathcal{F})$ at $e \in \mathrm{Exit}(\Gamma)$.

Assume that $\mathcal{F}$ factors through $\mathcal{P}r^L_{\mathrm{St}} \to \mathrm{St}$. For $e \in \mathrm{Exit}(\Gamma)$, the functor $\mathrm{ev}_e : \mathcal{H}(\Gamma, \mathcal{F}) \to \mathcal{F}(e)$ preserves all limits and colimits and thus admits a left adjoint ev^*_e.

Proposition 7.10. *Assume that for each vertex v of Γ, the spherical functor $F_v : \mathcal{V}_v \to \mathcal{N}$ classifying $\mathcal{F}$ at v is conservative, i.e. satisfies that $F_v(a) \simeq 0$ if and only if $a \simeq 0 \in \mathcal{V}_v$. Assume that the generic stalk $\mathcal{N}$ of $\mathcal{F}$ admits a compact generator b. The object*

$$\bigoplus_{\text{edges } e \in \mathrm{Exit}(\Gamma)} \mathrm{ev}^*_e(b) \in \mathcal{H}(\Gamma, \mathcal{F})$$

is a compact generator.

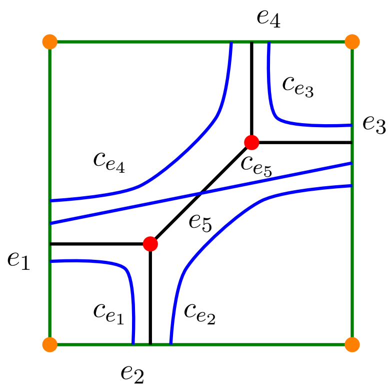
Proof. The fact that the right adjoint ev_e of ev^*_e preserves all colimits immediately implies that ev^*_e preserves compact objects, and thus that $\bigoplus_e \mathrm{ev}^*_e(b)$ is compact. The arguments in the proof of Proposition 6.13 imply that an object $X \in \mathcal{H}(\Gamma, \mathcal{F})$ is zero if and only if $\mathrm{ev}_e(X) \simeq 0$ for all edges $e \in \mathrm{Exit}(\Gamma)$. It thus immediate that $X \simeq 0$ if and only if $\mathrm{Map}(\bigoplus_{e \in \mathrm{Exit}(\Gamma)} \mathrm{ev}^*_e(b), X[i]) \simeq H_{-i}\left(\bigoplus_{e \in \mathrm{Exit}(\Gamma)} \mathrm{ev}_e(X)\right) \simeq 0$ for all $i \in \mathbb{Z}$. □

Compact generators via curves

Let $\mathbf{S}$ be a marked surface and Γ a spanning ribbon graph, such that every vertex has valency at least two. Note that every marked surface admits such a spanning graph, except for the disc with one boundary marked point and arbitrary punctures.

Let e be an edge of Γ, incident to the two vertices v_1, v_2. Consider the two curves c_1 and c_2 starting at e at v_1 and v_2 and composed of segments which always go one step counterclockwise around each vertex until they reach a boundary edge (in simple terms: one always keeps going right and stops when one hits the boundary). We can compose these two curves at e to an open pure matching curve c_e.

Example 7.11. The twice-punctured 4-gon with an ideal triangulation $\mathcal{T}$ with edges $e_1, \dots, e_5$ and associated pure matching curves $c_{e_1}, \dots, c_{e_5}$.



Theorem 7.12. *Let $\mathcal{F}$ be as in Construction 7.8 and let e be an edge of Γ. There exists an equivalence*[7] $M^b_{c_e} \simeq \mathrm{ev}^*_e(b) \in \mathcal{H}(\Gamma, \mathcal{F})$ *for all edges e of Γ.*

Proof sketch. (This proof will be skipped in the lecture.) Assume for simplicity that $\mathcal{N} = \mathcal{D}(R)$ for a dg-algebra R (in the general case, one can use the technology of linear ∞-categories). For δ a segment at v going from an edge e to an edge e' in the counterclockwise direction, consider the functor $\text{-} \otimes_R M^b_\delta : \mathcal{D}(R) \to \mathcal{L}(\Gamma, \mathcal{F})$. One can show, for example using the technology of Kan extensions, that its right adjoint $\mathrm{RHom}_R(M^b_\delta, \text{-})$ is equivalent to the functor

$$\widetilde{\mathrm{ev}_e} : \mathcal{L} \xrightarrow{\mathrm{ev}_e} \mathcal{V}^n_{F_v} \xrightarrow{\mathcal{F}(v \to e)} \mathcal{F}(e) = \mathcal{D}(R)$$

Similarly one has $\text{-} \otimes_R Z^b_e \dashv \mathrm{RHom}_R(Z^b_e, \text{-}) \simeq \mathrm{ev}_e$, where $\mathrm{ev}_e : \mathcal{L}(\Gamma, \mathcal{F}) \to \mathcal{F}(e) = \mathcal{D}(R)$ denotes the evaluation functor at e.

We now fix an edge e of Γ. Using that $\mathrm{RHom}_R(\text{-}, \text{-})$ preserves limits in the first entry, it follows that $\mathrm{RHom}_R(M^b_{c_e}, \text{-}) : \mathcal{L}(\Gamma, \mathcal{F}) \to \mathcal{D}(R)$ is equivalent to the limit of a diagram in $\mathrm{Fun}(\mathcal{L}(\Gamma, \mathcal{F}), \mathcal{D}(R))$ of the following form:

$$\begin{tikzcd}
 & \widetilde{\mathrm{ev}}_{e'} \arrow[dl] \arrow[dr] & & \widetilde{\mathrm{ev}}_{e} \arrow[dl] \arrow[dr] & & \widetilde{\mathrm{ev}}_{e} \arrow[dl] \arrow[dr] & & \widetilde{\mathrm{ev}}_{e''} \arrow[dl] \arrow[dr] & \\
\dots & & \mathrm{ev}_{e'} & & \mathrm{ev}_e & & \mathrm{ev}_{e''} & & \dots
\end{tikzcd}$$

Here e' is the edge of Γ where the first segment of c_1 (which, recall, is one half of c_e) end and e'' is the edge where the first segment of c_2 ends. Using that the restriction of $\widetilde{\mathrm{ev}_e}$ to $\mathcal{H}(\Gamma, \mathcal{F})$ is equivalent to ev_e, it becomes apparent that the restriction of the above diagram to $\mathrm{Fun}(\mathcal{H}(\Gamma, \mathcal{F}), \mathcal{D}(R))$ has limit ev_e. Passing to left adjoints yields $M^b_{c_e} \simeq \mathrm{ev}^*_e(b)$, as desired. $\square$

[7] This is not literally true in this form. In general, one must also account for the monodromy of $\mathcal{F}$ along c_1, by replacing b on the right side by $\mu(c_1)(b)$.

Example 7.13. Let $\mathbf{S}$ be the annulus with two boundary marked points and Γ be the spanning graph from (11) and $\mathcal{F}$ as in (12). Let e_1 and e_2 be the upper and lower internal edges of Γ. Then $M^k_{c_{e_1}}$ is equivalent to the global section X_2 from (13). The global section X_1 from (13) is equivalent to M^k_γ, with γ the curve in $\mathbf{S}$ going from one external edge to the other along the lower part of the ribbon graph. In terms of coherent sheaves, we have $X_1 \simeq O$, $X_2 \simeq M^k_{c_{e_1}} \simeq O(1)$ and $M^k_{c_{e_2}} \simeq O(2)$ in $\mathcal{H}(\Gamma, \mathcal{F}) \simeq \mathcal{D}(\mathrm{Coh}(\mathbb{P}^1))$. Theorem 7.12 thus boils down to the statement, that $O(1) \oplus O(2)$ is a compact generator (which is clear, since tensoring with $O(-1)$ is an autoequivalence of $\mathcal{D}(\mathrm{Coh}(\mathbb{P}^1))$ mapping $O(1) \oplus O(2)$ to $O \oplus O(1)$).

Example 7.14. Let $\mathcal{T}$ be an ideal triangulation of the 4-gon. Let $\mathcal{F}_\mathcal{T}$ be the perverse schober with $\mathcal{H}(\mathcal{F}, \mathcal{F}_\mathcal{T}) \simeq \mathcal{D}(\mathscr{G}_\mathcal{T})$, with $\mathscr{G}_\mathcal{T}$ the relative Ginzburg algebra depicted in Example 6.12. The five curves depicted in Example 7.11 give rise to five objects $M^{k[t_1]}_{c_{e_i}}$, which match the five projective $\mathscr{G}_\mathcal{T}$-modules associated to the five vertices of $Q_\mathcal{T}$.

More generally, for $\mathcal{T}$ any ideal n-angulation, one can show that the compact generator from Proposition 7.10 is equivalent to $\bigoplus_{e \in (Q_\mathcal{T})_0} e_i \mathscr{G}_\mathcal{T} = \mathscr{G}_\mathcal{T} \in \mathcal{D}(\mathscr{G}_\mathcal{T}) \simeq \mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T})$ and can thus be described in terms of pure matching curves (with $L = k[t_{n-2}]$.

Lecture 8: Geometric models II

One can generalize the construction of global sections of perverse schober via curves in a number of ways. For example, one can allow

i) matching curves composed of segments which wrap around a vertex of valency n by up to n steps. These matching curves are called non-pure.

ii) closed matching curves, i.e. maps $S^1 \to \mathbf{S} \backslash M$, equipped with an additional monodromy datum.

iii) matching curves with endpoints in the singularities of the perverse schober. These are called singular matching curves.

In the following, we describe the construction for curves as in iii). For more on the construction in the cases ii) and iii), we refer to [Chr21]. We then see how the Homs between the global sections from matching curves can be described in terms of intersections of the curves.

Objects from singular pure matching curves

We fix some marked surface with spanning graph Γ. A singular segment at a vertex v of Γ is a segment starting (or ending) at a vertex v and ending (or starting) at an edge e of Γ incident to v. Consider a Γ-parametrized perverse schober $\mathcal{F}$, which restricts near a vertex v of Γ to the diagram $\mathcal{F}_n(F)$ for some spherical functor $F : \mathcal{V} \to \mathcal{N}$. Let η_i be the singular segment beginning at v and ending at the i-th edge incident to v (in the chosen total order). The corresponding local section M^Q_{v,η_i}, where $Q \in \mathcal{V}$, is near v of the following form and vanishes everywhere else.

$$\begin{tikzcd}
\dots & & 0 & \\
0 & \left(Q \to F(Q) \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} \underbrace{F(Q)}_{(n-i)-th \text{ vertex}} \to 0 \to \dots \right)[1-i] \arrow[l] \arrow[ul] \arrow[u] \arrow[d] \arrow[r, "\varrho_i"] & F(Q) \\
& & 0 &
\end{tikzcd}$$

Exercise 7. Show that $M^Q_{v,\eta_i}(v) \simeq T^{i-1}_{\mathcal{V}^n_F}\left(M^Q_{v,\eta_1}(v)\right)$.

This section M^Q_{v,η_i} can be glued along one edge of Γ, where the non-singular endpoint of the segment η_i lies, with a local section with local value $L = F(Q)$. We can thus adapt the gluing process to produce global sections from singular matching curves with local value L lying in the image of the spherical functor.

Spherical objects from singular matching curves

We proceed with spelling out an example of the above construction for global sections arising from singular matching curves.

Example 8.1. Consider the ideal triangulation $\mathcal{T} = \Gamma$ of the twice punctured 4-gon from Example 7.5. The $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_\mathcal{T}$ is given as follows:

$$\begin{tikzcd}
 & \mathcal{D}(k[t_1]) & & & \\
\mathcal{D}(k[t_1]) & \mathcal{V}^3_{\phi^*} \arrow[l, "\varrho_2"] \arrow[u, "\varrho_1"] \arrow[r, "\varrho_3"] & \mathcal{D}(k[t_1]) & \mathcal{V}^3_{\phi^*} \arrow[l, "\varrho_2"] \arrow[r, "\varrho_1"] \arrow[d, "\varrho_3"] & \mathcal{D}(k[t_1]) \\
 & & & \mathcal{D}(k[t_1]) &
\end{tikzcd}$$

Let e be the unique internal edge of $\mathcal{T}$. We consider e as a singular matching curve, composed of the singular segments η_3, η_2 at the two punctures v_1, v_2 incident to e. We let $Q = k \in \mathcal{D}(k) = \mathcal{N}$ and thus have associated sections $M^k_{v_1,\eta_3}, M^k_{v_2,\eta_2}$ and $Z^{\phi^*(k)}_e$. The global section M^k_e is the pushout of $M^k_{v_1,\eta_3}$, $M^k_{v_2,\eta_2}$ and $Z^{\phi^*(k)}_e$, and given as follows

$$\begin{tikzcd}
 & 0 & & & \\
0 & (k \to 0 \to 0)[-2] \arrow[l] \arrow[u] \arrow[r] & \phi^*(k) & (k \to \phi^*(k) \to 0)[-1] \arrow[l] \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

The $\mathcal{G}_\mathcal{T}$-module $M^{\phi(k)}_e$ admits the following algebraic interpretation. Let i be a vertex of the quiver $Q_\mathcal{T}$ (which, recall, are just the edges of $\mathcal{T}$) and $P_i = e_i\mathcal{G}_\mathcal{T}$ be the corresponding projective module. Using that $P_i \simeq \mathrm{ev}^*_i(k[t_1])$, we get

$$\mathrm{Ext}^*(P_i, M^{\phi^*(k)}_e) \simeq \mathrm{Ext}^*_{\mathcal{D}(k[t_1])}(k[t_1], \mathrm{ev}_e(M^{\phi^*(k)}_e)) \simeq \begin{cases} \mathrm{Ext}^*(k[t_1], \phi^*(k)) \simeq k & i = e \\ 0 & \text{else .} \end{cases}$$

It follows that $M^{\phi^*(k)}_e$ is the object arising from the simple $\mathcal{G}_\mathcal{T}$-module associated to the vertex e of $Q_\mathcal{T}$.

Let us compute the endomorphisms of $M^{\phi^*(k)}_e$. For that, we consider the derived Hom-functor

$$\mathrm{RHom}(\text{-},\text{-}) = \mathrm{RHom}_{\mathcal{L}(\mathcal{T},\mathcal{F}_\mathcal{T})}(\text{-},\text{-}) : \mathcal{L}(\mathcal{T},\mathcal{F}_\mathcal{T})^{\mathrm{op}} \times \mathcal{L}(\mathcal{T},\mathcal{F}_\mathcal{T}) \to \mathcal{D}(k)$$

which preserves limits in both entries. We also write $\mathrm{REnd}(X)$ for $\mathrm{RHom}(X,X)$.

Remark 8.2. By the derived Hom-functor, we actually mean the linear morphism object functor for k-linear ∞-categories, whose definition we recall in Appendix A.

Plugging into $\mathrm{RHom}(\text{-}, M_e^{\phi^*(k)})$ the pushout diagram defining $M_e^{\phi^*(k)}$, we obtain the pullback diagram

$$\begin{tikzcd}
\mathrm{RHom}(M_e^{\phi^*(k)}, M_e^{\phi^*(k)}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{RHom}(M_{v_1,\eta_3}^{k}, M_e^{\phi^*(k)}) \arrow[d] \\
\mathrm{RHom}(M_{v_2,\eta_2}^{k}, M_e^{\phi^*(k)}) \arrow[r] & \mathrm{RHom}(Z_e^{\phi^*(k)}, M_e^{\phi^*(k)})
\end{tikzcd} \tag{14}$$

which allows us to reduce the computation of $\mathrm{REnd}(M_e^{\phi^*(k)}) = \mathrm{RHom}(M_e^{\phi^*(k)}, M_e^{\phi^*(k)})$ to local computations.

Lemma 8.3. *Let Γ be a ribbon graph, v a vertex and e an edge of Γ. Let $\mathcal{F}$ be a Γ-parametrized perverse schober and γ a matching curve. For each segment δ at v, one has*

$$\mathrm{RHom}_{\mathcal{L}(\Gamma,\mathcal{F})}(M_{v,\delta}^{L}, M_\gamma^{L'}) \simeq \mathrm{RHom}_{\mathcal{F}(v)}(M_{v,\delta}^{L}(v), M_\gamma^{L'}(v))\,.$$

Similarly, one has

$$\mathrm{RHom}_{\mathcal{L}(\Gamma,\mathcal{F})}(Z_e^{L}, M_\gamma^{L'}) \simeq \mathrm{RHom}_{\mathcal{F}(e)}(L, M_\gamma^{L'}(e))\,.$$

Proof. This follows from the fact that $M_{v,\delta}^{L}, Z_e^{L}$ are p-relative left Kan extensions of their restrictions along $\Delta^0 \xrightarrow{x} \mathrm{Exit}(\Gamma)$, with $x = v, e$ and p the Grothendieck construction of $\mathcal{F}$. $\square$

In the example, we have

$$\begin{aligned}
\mathrm{RHom}(M_{v_1,\eta_3}^{k}(v), M_e^{\phi^*(k)}(v)) &\simeq \mathrm{REnd}(k \to 0 \to 0) \simeq k \\
\mathrm{RHom}(M_{v_2,\eta_2}^{k}(v), M_e^{\phi^*(k)}(v)) &\simeq \mathrm{REnd}(k \to \phi^*(k) \to 0) \simeq k \\
\mathrm{RHom}(Z_e^{\phi^*(k)}(e), M_e^{\phi^*(k)}(e)) &\simeq \mathrm{REnd}(\phi^*(k)) \simeq \mathrm{H}^*(S^2; k) \simeq k \oplus k[-2]
\end{aligned}$$

The biCartesian square (14) thus takes the form

$$\begin{tikzcd}
\mathrm{REnd}(M_e^{\phi^*(k)}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & k \arrow[d, "(\mathrm{id},0)"] \\
k \arrow[r, "(\mathrm{id},0)"] & k \oplus k[-2]
\end{tikzcd}$$

It follows that $\mathrm{REnd}(M_e^{\phi^*(k)}) \simeq k \oplus k[-3] \simeq H^*(S^3, k)$. We thus call $M_e^{\phi^*(k)}$ a 3-spherical object. Note that if we just naively try to find all non-zero natural endotransformations of $M_e^{\phi^*(k)}$, we get $k \subset \mathrm{REnd}(M_e^{\phi^*(k)})$. The endomorphisms corresponding to $k[-3] \subset \mathrm{REnd}(M_e^{\phi^*(k)})$ yield natural endotransformations of the global section $M_e^{\phi^*(k)}$ which evaluate everywhere to zero! The non-trivial data is hidden in a pair of null-homotopies.

Endomorphisms of global sections from matching curves

The above example can be generalized as follows.

Proposition 8.4 (Special case of [Chr21, Thm. 6.5])**.** *Let $\mathcal{T}$ be an ideal n-angulation of a marked surface $\mathbf{S}$ and let γ be a pure matching curve without self-intersections.*

- *If both endpoints γ lie on $\partial\mathbf{S}\backslash M$, then $\mathrm{REnd}(M_\gamma^{L}) \simeq \mathrm{REnd}(L)$ for all $L \in \mathcal{D}(k[t_{n-2}])$. If $L = \phi^*(k)$, then $\mathrm{REnd}(L) \simeq H^*(S^{n-1}; k) \simeq k \oplus k[1-n]$, meaning that $M_\gamma^{\phi^*(k)}$ is an $(n-1)$-spherical object.*

- *If γ has a single endpoint at a vertex of $\mathcal{T}$ and a single endpoint on $\partial \mathbf{S} \backslash M$, then* $\operatorname{REnd}(M_\gamma^{\phi^*(k)}) \simeq k$*, meaning that $M_\gamma^{\phi^*(k)}$ is an exceptional object.*

- *If both endpoints of γ are vertices of $\mathcal{T}$, then* $\operatorname{REnd}(M_\gamma^{\phi^*(k)}) \simeq k \oplus k[-n]$*, meaning that $M_\gamma^{\phi^*(k)}$ is an n-spherical object.*

Corollary 8.5. *Assume that k is a field. Let γ be a pure matching curve without self-intersections and $L \in \mathcal{D}(k[t_{n-2}])$ with $H_0 \operatorname{REnd}(L) \simeq k$. Then the global section $M_\gamma^L \in \mathcal{H}(\mathcal{T}, \mathcal{F}_\mathcal{T})$ is indecomposable.*

Proof. By Proposition 8.4, we have $H_0 \operatorname{REnd}(M_\gamma^L) \simeq k$, which is a local ring. It follows that M_γ^L is indecomposable. □

Corollary 8.5 can be generalized to the statement that the global sections M_γ^L associated to an arbitrary pure matching curve and L with $H_0 \operatorname{REnd}(L) \simeq k$ is indecomposable, though the proof becomes a little bit more involved/interesting.

Homs and the different types of intersections.

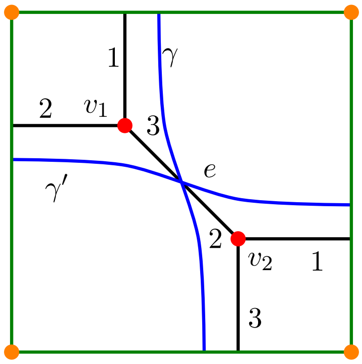
We have already encountered directed boundary intersections of matching curves in Lecture 7, see Definition 7.1 and that these, at least on the disc, give rise to morphisms between the associated global sections. We consider the following two further types of intersections.

Definition 8.6. Let γ, γ' be matching curves in a marked surface with a spanning graph Γ.

- A crossing of γ and γ' is an intersection of γ and γ' away from their endpoints.
- A singular intersection fo γ and γ' is an intersection of γ and γ' at their endpoints in vertices of Γ.

Crossing and singular intersection also give rise to morphism between sections associated to pure matching curves.

Example 8.7. We again consider the $\mathcal{T}$-parametrized perverse schober $\mathcal{F}_\mathcal{T}$ on the twice-punctured 4-gon from Example 8.1. Consider the following two pure matching curves γ, γ' with a crossing.



The integer labeling of the edges at the two vertices denote their position in the total orders. The curve γ consists of the segments $\delta_{v_1,1}$ and $\delta_{v_2,3}$. The curve γ' consists of the the segments $\delta_{v_1,3}$ and $\delta_{v_2,2}$. For any two $L, L' \in \mathcal{N} = \mathcal{D}(k[t_1])$, we thus have pullback diagram

$$\begin{tikzcd} \operatorname{RHom}(M_\gamma^L, M_{\gamma'}^{L'}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{RHom}(M_{v_1,\delta_1}^L, M_{\gamma'}^{L'}) \arrow[d] \\ \operatorname{RHom}(M_{v_2,\delta_3}^L, M_{\gamma'}^{L'}) \arrow[r] & \operatorname{RHom}(Z_e^L, M_{\gamma'}^{L'}) \end{tikzcd} \qquad \begin{tikzcd} \operatorname{RHom}(M_{\gamma'}^{L'}, M_\gamma^L) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{RHom}(M_{v_1,\delta_3}^{L'}, M_\gamma^L) \arrow[d] \\ \operatorname{RHom}(M_{v_2,\delta_2}^{L'}, M_\gamma^L) \arrow[r] & \operatorname{RHom}(Z_e^{L'}, M_\gamma^L) \end{tikzcd}$$

Using Lemma 8.3, we find:

$$
\begin{aligned}
\mathrm{RHom}(Z_e^L, M_{\gamma'}^{L'}) &\simeq \mathrm{RHom}(L, L') \\
\mathrm{RHom}(Z_e^{L'}, M_{\gamma}^{L}) &\simeq \mathrm{RHom}(L', L) \\
\mathrm{RHom}(M_{v_1,\delta_1}^L, M_{\gamma'}^{L'}) &\simeq \mathrm{RHom}(M_{v_1,\delta_1}^L, M_{v_1,\delta_3}^{L'}) \simeq \mathrm{RHom}(\varsigma_1(L), \varsigma_3(L')) \simeq \mathrm{Mor}(L, L') \\
\mathrm{RHom}(M_{v_2,\delta_3}^L, M_{\gamma'}^{L'}) &\simeq \mathrm{RHom}(M_{v_2,\delta_3}^L, M_{v_2,\delta_2}^{L'}) \simeq \mathrm{RHom}(\varsigma_3(L), \varsigma_2(L')) \simeq \mathrm{Mor}(L, L') \\
\mathrm{RHom}(M_{v_1,\delta_3}^{L'}, M_{\gamma}^{L}) &\simeq \mathrm{RHom}(M_{v_1,\delta_3}^{L'}, M_{v_1,\delta_1}^{L}) \simeq \mathrm{RHom}(\varsigma_3(L'), \varsigma_1(L)) \simeq 0 \\
\mathrm{RHom}(M_{v_2,\delta_2}^{L'}, M_{\gamma}^{L}) &\simeq \mathrm{RHom}(M_{v_2,\delta_2}^{L'}, M_{v_2,\delta_3}^{L}) \simeq 0
\end{aligned}
$$

The pullback diagrams are thus equivalent to

$$
\begin{tikzcd}
\mathrm{RHom}(M_\gamma^L, M_{\gamma'}^{L'}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{RHom}(L, L') \arrow[d, "\simeq"] \\
\mathrm{RHom}(L, L') \arrow[r, "\simeq"] & \mathrm{RHom}(L, L')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{RHom}(M_{\gamma'}^{L'}, M_{\gamma}^{L}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{RHom}(L', L)
\end{tikzcd}
$$

showing that $\mathrm{RHom}(M_\gamma^L, M_{\gamma'}^{L'}) \simeq \mathrm{RHom}(L, L')$ and $\mathrm{RHom}(M_{\gamma'}^{L'}, M_\gamma^L) \simeq \mathrm{RHom}(L', L)[-1]$. The crossing of γ and γ' thus gives rise to morphisms in both directions.

In total, one can show the following.

Theorem 8.8 ([Chr21]). *Let $\mathcal{T}$ be an ideal n-angulation of an oriented marked surface and γ, γ' be two distinct open pure matching curves. Let $L, L' \in \mathcal{D}(k[t_{n-2}])$ and consider the associated global sections $M_\gamma^L, M_{\gamma'}^{L'} \in \mathcal{H}(\mathcal{T}, \mathcal{F}_{\mathcal{T}})$. Then $\mathrm{RHom}(M_\gamma^L, M_{\gamma'}^{L'}) \in \mathcal{D}(k)$ is the direct sum of the following k-modules:*

- *For each directed boundary intersection from γ to γ', there is direct summand* $\mathrm{RHom}(L, L')$.
- *For each crossing of γ and γ', there is a direct summand* $\mathrm{RHom}(L, L')$ *or* $\mathrm{RHom}(L, L')[-1]$.
- *If γ and γ' have singular intersections, then $L = L' = \phi^*(k)$ and there is a direct summand given by $k[-m]$, with some $0 \leq m \leq n$, for each singular intersection of γ and γ'.*

An analogue of the above theorem, incorporating the monodromy data of the perverse schober (such as gradings), should hold for arbitrary[8] perverse schobers and open pure matching curves. An analogue of the above theorem for non-pure matching curves fails to holds, unless $L = L' = \phi^*(k)$, see [Chr21].

A Appendix: linear ∞-categories

The ∞-category $\mathcal{P}r^L$ admits a symmetric monoidal structure; the monoidal product $\otimes$ is defined by the property that a functor $\mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ is the same thing as a functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ preserving colimits in both entries. We refer to [Lur17, Section 4.8] for background. The commutative algebra objects in $\mathcal{P}r^L$ can hence be identified with presentable, symmetric monoidal ∞-categories $\mathcal{C}$, whose monoidal product $\text{-} \otimes \text{-} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves colimits in both entries. An example of such a symmetric monoidal ∞-category is the derived ∞-category of a commutative dg-algebra R, for example $R = k$ a commutative ring. Using the general machinery of ∞-categorical modules of [Lur17, Section 3.3], one can thus consider left modules over $\mathcal{D}(R)$ in

[8]Assuming however that the vertices of the parametrizing ribbon graph have valency ≥ 2. The case with 1-valent vertices is more complicated.

$\mathcal{P}r^L$, which are called R-linear ∞-categories. The module action equips an R-linear ∞-category $\mathcal{C}$ with a tensor map $\text{-} \otimes \text{-} : \mathcal{D}(R) \times \mathcal{C} \to \mathcal{C}$ which preserves colimits in both entries. Morphisms of left $\mathcal{D}(R)$-modules are called R-linear functors. The forgetful functor $\mathrm{LinCat}_{\mathcal{D}(R)} \to \mathrm{Cat}_\infty$ from the ∞-category of R-linear ∞-categories preserves all limits.

Example A.1. The derived ∞-categories of k-linear dg-categories are k-linear ∞-categories, see [Coh13].

Example A.2. Let A, B be k-linear dg-algebras. There exists an equivalence of ∞-categories

$$\mathrm{Lin}_k(\mathcal{D}(A), \mathcal{D}(B)) \simeq \mathcal{D}(A^{\mathrm{op}} \otimes_k B)$$

between the ∞-category of k-linear functors and the ∞-category of A-B-bimodules, see [Lur17, 4.8.4.1].

Definition A.3 ([Lur17, 4.2.1.28])**.** Let $\mathcal{C}$ be an R-linear ∞-category and $X, Y \in \mathcal{C}$. A morphism object $\mathrm{Mor}_{\mathcal{C}}(X,Y) \in \mathcal{D}(R)$ is an R-module equipped with a morphism in $\mathcal{C}$

$$\alpha : \mathrm{Mor}_{\mathcal{C}}(X,Y) \otimes_R X \to Y,$$

such that for every object $C \in \mathrm{RMod}_R$ composition with α induces an equivalence of spaces

$$\mathrm{Map}_{\mathcal{D}(R)}(C, \mathrm{Mor}_{\mathcal{C}}(X,Y)) \to \mathrm{Map}_{\mathcal{C}}(C \otimes X, Y)\,.$$

Morphism objects in R-linear ∞-categories always exist and satisfy

$$\mathrm{H}_i \, \mathrm{Mor}_{\mathcal{C}}(X,Y) \simeq \pi_0 \, \mathrm{Map}_{\mathcal{C}}(X[i], Y) =: \mathrm{Ext}^{-i}(X,Y)\,.$$

The formation of morphism objects forms a functor

$$\mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-}) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}(R)$$

which preserves limits in both entries. For $X \in \mathcal{C}$, the functor $\mathrm{Mor}_{\mathcal{C}}(X, \text{-})$ is right adjoint to the functor $\text{-} \otimes_R X : \mathcal{D}(R) \to \mathcal{C}$. Given an R-linear functor $F : \mathcal{C} \to \mathcal{D}$, there is an natural transformation $F : \mathrm{Mor}_{\mathcal{C}}(\text{-},\text{-}) \to \mathrm{Mor}_{\mathcal{D}}(F(\text{-}), F(\text{-}))$ restricting on H_0 to applying F to the morphisms.

Lemma A.4. *Let $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ be an adjunction between R-linear ∞-categories, such that F and G are R-linear functors. Let $u : \mathrm{id}_{\mathcal{C}} \to GF$ be the unit of $F \dashv G$. Then the morphism in $\mathcal{D}(R)$*

$$\mathrm{Mor}_{\mathcal{D}}(F(X), Y) \xrightarrow{G} \mathrm{Mor}_{\mathcal{C}}(GF(X), F(Y)) \xrightarrow{u} \mathrm{Mor}_{\mathcal{C}}(X, F(Y)) \tag{15}$$

is an equivalence in $\mathcal{D}(R)$.

Proof. Applying H_i to the morphism (15) yields an equivalence for all $i \in \mathbb{Z}$ by the adjunctions $F \dashv G$. It follows that (15) is a quasi-isomorphism, and thus an equivalence in $\mathcal{D}(R)$. □

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