Perverse schobers and exceptional collections

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The goal of these notes is to explain how k-linear stable ∞-categories with full exceptional collections can be studied using the formalism of parametrized perverse schobers.

In Section 1, we begin with a brief summary of the notion of a perverse schober parametrized by a ribbon graph, and then recall some properties of its induction functors.

In Section 2, we will recall the construction of the directed category on a subset of objects of an ∞-category. We will then show that any directed category on a finite set of spherical objects arises as the global sections of a perverse schober on the disc. The parametrizing spanning graph will be given by a linear graph with a single external edge.

In the final Section 3, given a perverse schober parametrized by a linear graph with a single external edge, we will construct a new perverse schober parametrized by the same graph, together with two equivalences between their ∞-categories of global sections. These two equivalences will have the property of mapping left induced object to right induced objects (or vice versa). In the case of a perverse schober arising from a collection of spherical objects, they will thus map the standard exceptional collection to the costandard coexceptional collection, or vice versa, reminiscent of the Ringel duality in the theory of highest weight abelian categories.

Prerequisites: We will assume familiarity with the theory of stable ∞-categories and the theory of dg categories.

Contents

1 Recollections on perverse schobers

1.1 Marked surfaces and spanning graphs

Definition 1.1. A marked surface $(\mathbf{S}, M)$ consists of compact oriented topological surface $\mathbf{S}$ with non-empty boundary $\mathbf{S}$ and a finite set of marked points $M \subset \mathbf{S}$, such that every boundary component contains at least one marked point. We typically just write $\mathbf{S}$ for the marked surface $(\mathbf{S}, M)$.

In the later parts of these notes, we will mostly be interested in the disc, considered as a marked surface with one or two marked points, also called the 1-gon and 2-gon.

By a graph $\mathbf{G}$, we will mean a graph with a finite set of vertices and edges. We allow external edges, meaning edges which are incident once to only a single vertex. Each internal edge of a ribbon graph consists of two halfedges, lying at the two vertices incident to the edge. For simplicity, we will not allow loops in graphs, meaning edges which are incident to the same vertex twice.

The geometric realization $|\mathbf{G}|$ of a graph $\mathbf{G}$ is the corresponding topological space, obtained by gluing together an interval for every edge along the vertices.

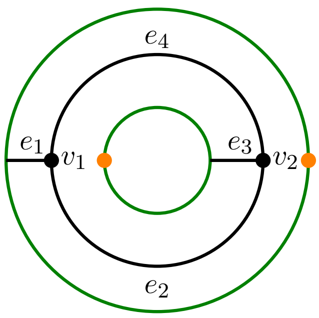
Definition 1.2. We call a graph $\mathbf{G}$ a spanning graph of a marked surface $\mathbf{S}$ if it is equipped with an embedding $i\colon |\mathbf{G}| \subset \mathbf{S}\backslash M$ satisfying that

- i is a homotopy equivalence,
- only the external endpoints of the external edges intersect the boundary $\partial\mathbf{S}$, and
- ι induces a bijection between the set of external edges of $\mathbf{G}$ and the connected components of $\partial\mathbf{S}\backslash M$.

Remark 1.3. A ribbon graph is a graph $\mathbf{G}$ equipped with a cyclic order on the set of the incident half-edges at every vertex v of $\mathbf{G}$.

If $\mathbf{G}$ is a spanning graph of a marked surface $\mathbf{S}$, then it inherits a canonical ribbon graph structure, via the counterclockwise order induced by the orientation of $\mathbf{S}$.

Example 1.4. The annulus (in green) with two marked points (in orange) together with a spanning graph (in black).

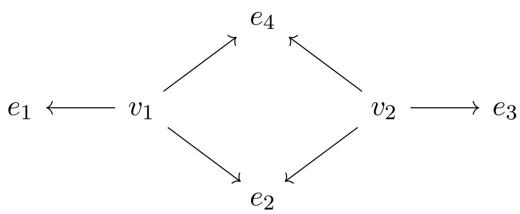


1.2 Exit paths and global sections

Definition 1.5. The exit path category $\mathrm{Exit}(\mathbf{G}) \in \mathrm{Cat}_\infty$ of a graph $\mathbf{G}$ is defined as the nerve of the 1-category with

- objects the vertices and edges of $\mathbf{G}$ and
- non-identity morphisms of the form $v \to e$ with v a vertex incident to an edge e.

Example 1.6. The exit path category of the trivalent spanning graph from Example 1.4 can be depicted as follows:



Remark 1.7. The exit path category of a graph $\mathbf{G}$ is 1-skeletal, and hence for any ∞-category $\mathcal{C}$ there exists a bijection

$$\pi_0 \operatorname{Fun}(\operatorname{Exit}(\mathbf{G}), \mathcal{C}) \simeq \operatorname{Hom}_{\operatorname{Set}_\Delta}(\operatorname{Exit}(\mathbf{G}), \mathcal{C}) \simeq \operatorname{Hom}_{\operatorname{Fun}((\Delta^{\leq 1})^{\mathrm{op}}, \operatorname{Set})}(\operatorname{tr}_{\leq 1}(\operatorname{Exit}(\mathbf{G})), \operatorname{tr}_{\leq 1}(\mathcal{C}))\,,$$

where $\operatorname{tr}_{\leq 1}(\mathcal{C}) \in \operatorname{Fun}((\Delta^{\leq 1})^{\mathrm{op}}, \operatorname{Set})$ is the 1-truncation, i.e. the pullback along $(\Delta^{\leq 1})^{\mathrm{op}} \subset \Delta^{\mathrm{op}}$. Informally, this means that an ∞-functor $\operatorname{Exit}(\mathbf{G}) \to \mathcal{C}$ is uniquely specified by assignments on objects and morphisms (an no further 'higher' data).

Remark 1.8. Let St denote the ∞-category of (small) stable ∞-categories. We can consider ribbon graphs as stratified spaces with the 0-strata given by the vertices and the 1-strata by the (open) edges. The datum of a functor $\operatorname{Exit}(\mathbf{G}) \to \operatorname{St}$ can be shown to be equivalent to the datum of a St-valued constructible sheaf on $\mathbf{G}$, see for instance [PT22].

Motivated by the above remark, we introduce the following terminology:

Definition 1.9. Let $\mathcal{F}\colon \operatorname{Exit}(\mathbf{G}) \to \operatorname{St}$ be a functor. We denote the limit of $\mathcal{F}$ by

$$R\Gamma(\mathbf{G}, \mathcal{F}) \coloneqq \lim \mathcal{F} \in \operatorname{St}$$

and call this stable ∞-category the ∞-category of global sections of $\mathcal{F}$.

We note that the forgetful functor $\operatorname{St} \to \operatorname{Cat}_\infty$ preserves limits.

1.3 Perverse schobers

For $n \geq 1$, we denote by Sp_n the n-spider, given by the ribbon graph with a single vertex v and n incident external edges $e_1, \dots, e_n$.

Definition 1.10 ([CHQ23])**.** Let $n \geq 1$. A perverse schober on the n-spider consists of the following data:

(1) If $n = 1$, a spherical functor between stable ∞-categories

$$F\colon \mathcal{V} \to \mathcal{N}$$

meaning that F admits a right adjoint G, such that the twist functor $C_{\mathcal{V}} = \operatorname{cof}(\operatorname{id}_{\mathcal{V}} \xrightarrow{\text{unit}} GF) \in \operatorname{Fun}(\mathcal{V}, \mathcal{V})$ and cotwist functor $C_{\mathcal{N}} = \operatorname{fib}(FG \xrightarrow{\text{counit}} \operatorname{id}_{\mathcal{N}}) \in \operatorname{Fun}(\mathcal{N}, \mathcal{N})$ are autoequivalences.

(2) If $n \geq 2$, a collection of functors of stable ∞-categories

$$(F_i\colon \mathcal{V}^n \longleftrightarrow \mathcal{N}_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$$

satisfying that

(a) F_i admits adjoints $F_i \dashv G_i \dashv H_i$,

(b) G_i is fully faithful, which is equivalent to $F_i G_i \simeq \mathrm{id}_{\mathcal{N}_i}$ via the counit,

(c) $F_i \circ G_{i-1}$ is an equivalence of ∞-categories,

(d) $F_i \circ G_j \simeq 0$ if $j \neq i, i-1$,

(e) $\mathrm{fib}(H_{i-1}) = \mathrm{fib}(F_i)$ as full subcategories of $\mathcal{V}^n$.

We remark that condition (e) implies condition (c).

Note that a collection of functors as above is the same data as a functor $\mathrm{Exit}(\mathrm{Sp}_n) \to \mathrm{St}$, with Sp_n the n-spider, mapping $v \to e_i$ to F_i.

Remark 1.11. Let $n \geq 2$ and $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}_n) \to \mathrm{St}$ a perverse schober on the n-spider. Using condition (e), one can show that

$$\mathcal{F}(v \to e_i)^L \simeq \mathcal{F}(v \to e_{i-1})^R \circ \mathcal{F}(v \to e_{i-1}) \circ \mathcal{F}(v \to e_i)^L .$$

Stated informally, this means that left induction (see also below) from e_i to v yields the same objects as right induction from e_{i+1} to v.

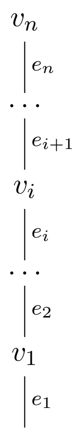
Definition 1.12. A functor $\mathcal{F}\colon \mathrm{Exit}(\mathbf{G}) \to \mathrm{St}$ is called a $\mathbf{G}$-parametrized perverse schober if for each vertex v of valency n of $\mathbf{G}$, with corresponding inclusion $\mathrm{Exit}(\mathrm{Sp}_n) \subset \mathrm{Exit}(\mathbf{G})$, the restriction $\mathcal{F}|_{\mathrm{Exit}(\mathrm{Sp}_n)}$ defines a perverse schober on the n-spider in the sense of Definition 1.10

1.4 Induction

Given a $\mathbf{G}$-parametrized perverse schober $\mathcal{F}$, the limit diagram for $R\Gamma(\mathbf{G}, \mathcal{F})$ supplies for each vertex v of $\mathbf{G}$ a functor $R\Gamma(\mathbf{G}, \mathcal{F}) \to \mathcal{F}(v)$, which we denote by ev_v. One can show that ev_v admits left and right adjoint functors [Chr25].

Definition 1.13. Let v be a vertex of $\mathbf{G}$. The left induction functor $\mathrm{ind}_v^L\colon \mathcal{F}(v) \to R\Gamma(\mathbf{G}, \mathcal{F})$ from v is defined as the left adjoint of ev_v. Similarly, the right induction functor $\mathrm{ind}_v^R\colon \mathcal{F}(v) \to R\Gamma(\mathbf{G}, \mathcal{F})$ is defined as the right adjoint of ev_v.

For any $n \geq 1$, we let $\mathbf{G}^{\downarrow,m}$ be the linear graph on n-vertices with one external edge, which we depict as follows:



Note that $\mathbf{G}^{\downarrow,m}$ is for any $n \geq 1$ a spanning graph of the 1-gon.

Proposition 4.12 together with the proof of Lemma 4.20 in [Chr25] imply the following:

Lemma 1.14 ([Chr25])**.** *Let $\mathcal{F}$ be a $\mathbf{G}^{\downarrow,m}$-parametrized perverse schober.*

(1) The functors $\mathrm{ind}^L_{v_n}, \mathrm{ind}^R_{v_n} : \mathcal{F}(v_n) \to R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F})$ *are fully faithful.*

(2) Then for all $1 \leq i < n$ *the composite functors*

$$\mathrm{fib}(\mathcal{F}(v_i \to e_{i+1})) \subset \mathcal{F}(v_i) \xrightarrow{\mathrm{ind}^L_{v_i}} R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F})$$

and

$$\mathrm{fib}(\mathcal{F}(v_i \to e_{i+1})) \subset \mathcal{F}(v_i) \xrightarrow{\mathrm{ind}^R_{v_i}} R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F})$$

are fully faithful.

(3) Let $X \in \mathrm{fib}(\mathcal{F}(v_i \to e_{i+1}))$ *and* $Y = \mathcal{F}(v_i \to e_i)(X) \in \mathcal{F}(e_i)$*. Then there exists an equivalence of* ∞*-categories* $\mathcal{F}(e_i) \simeq \mathcal{F}(e_1)$*, denoted* $\mathcal{F}^{\to}(\gamma)$[1]*, such that*

$$\mathrm{ev}_{e_1}(\mathrm{ind}^L_{v_i}(X)) \simeq \mathcal{F}^{\to}(\gamma)(Y) \in \mathcal{F}(e_1)\,.$$

2 Perverse schobers for directed categories

2.1 The directed subcategory on an ordered collection of objects

We fix a field k.

Definition 2.1. Let C be a k-linear dg category and $X_1, \ldots, X_m \in C$ a finite ordered collection of (not necessarily distinct) objects. We define $\hat{C}(X_1, \ldots, X_m)$ as the dg category with

- objects $\hat{X}_1, \ldots, \hat{X}_m$ and
- $$\mathrm{RHom}_{\hat{C}(X_1,\ldots,X_m)}(\hat{X}_i, \hat{X}_j) = \begin{cases} \mathrm{RHom}_C(X_i, X_j) & i < j \\ k & i = j \\ 0 & \text{else.} \end{cases}$$
- composition of morphisms induced by the composition of morphisms in C.

We note that there is a canonical dg-functor $\hat{C}(X_1, \ldots, X_m) \to C$.

Remark 2.2. An A_∞-categorical version of the above definition was given by Seidel in [Sei08, Section I.(5n)], called the directed category construction.

Definition 2.3. (1) We call a dg category A with a directed set of objects $Y_1, \ldots, Y_m$ upper triangular unipotent if $\mathrm{RHom}_A(Y_i, Y_j) \simeq 0$ for $j < i$ and $\mathrm{RHom}_A(Y_i, Y_i) \simeq k$ as chain complexes.

(2) We call a dg category A with a directed set of objects $Y_1, \ldots, Y_{m'}$ upper triangular quasi-unipotent if $\mathrm{RHom}_A(Y_i, Y_j) \simeq 0$ for $j < i$ and $\mathrm{RHom}_A(Y_i, Y_i) \simeq k$ in $D(k)$.

Remark 2.4. The directed subcategory comes with a canonical dg functor $\hat{C}(X_1, \ldots, X_m) \to C$, mapping $\hat{X}_i$ to X_i.

Given a dg functor $f : A \to C$ with A upper triangular unipotent, if it factors through the dg functor $\hat{C}(X_1, \ldots, X_m) \to C$, then there is a unique such factorization. This is the case if the image of A is contained in $\{X_1, \ldots, X_m\}$ and f preserves the respective orders on objects.

[1] Here γ refers to the curve starting at e_i and going downwards to e_1, always 'turning left' at the 2-valent vertices for left induction, and always 'turning right' for right induction. The equivalence $\mathcal{F}^{\to}(\gamma)$ is the transport equivalence of $\mathcal{F}$ along γ.

Lemma 2.5. *Any upper triangular quasi-unipotent dg category C admits a quasi-equivalence $C' \to C$ from a cofibrant upper triangular unipotent dg category.*

Proof. The dg functor $\hat{C}(Y_1, \dots, Y_m) \to C$ is a quasi equivalence from a upper triangular unipotent dg category. We further can cofibrantly $\hat{C}(Y_1, \dots, Y_m)$ via a cofibrant upper triangular unipotent dg category, see p.14 in [Kel06]. □

Lemma 2.6. *Let A be an upper triangular quasi-unipotent dg category with objects $Y_1, \dots, Y_m$ and $f\colon A \to C$ a dg functor inducing a quasi-equivalence $\mathrm{RHom}_A(Y_i, Y_j) \simeq \mathrm{RHom}_C(f(Y_i), f(Y_j))$ for all $i < j$. Then there exists a zig-zag of quasi-equivalences between A and $\hat{C}(f(Y_1), \dots, f(Y_m))$.*

Proof. Let $A' \to A$ be a unipotent replacement. Then the composite functor $A' \to C$ lifts by Remark 2.4 to a quasi-equivalence $A' \to \hat{C}(f(Y_1), \dots, f(Y_m))$. □

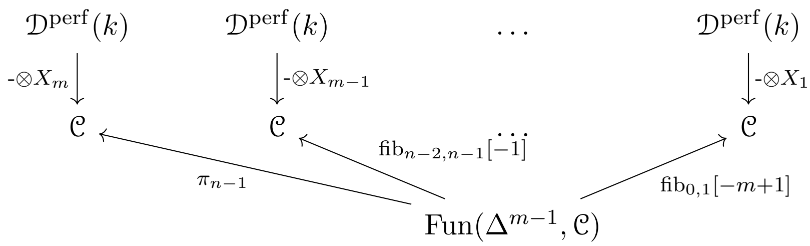
Let $\mathcal{C}$ be a k-linear, idempotent complete stable ∞-category and $X_1, \dots, X_m \in \mathcal{C}$ a finite ordered collection of (not necessarily distinct) objects. Then there exists a dg category C, unique up to Morita equivalence, together with an equivalence of k-linear ∞-categories $\mathcal{D}^{\mathrm{perf}}(C) \simeq \mathcal{C}$. Let $Y_1, \dots, Y_m \in \mathrm{Perf}(C)$ be a collection of dg modules, whose image in $\mathcal{D}^{\mathrm{perf}}(C) \simeq \mathcal{C}$ is equivalent to $X_1, \dots, X_m$. We further choose a collection of fibrant-cofibrant dg modules $Z_1, \dots, Z_m \in \mathrm{Perf}(\hat{C}(Y_1, \dots, Y_m))$, such that $Z_i \simeq \hat{Y}_i \in \mathrm{dgMod}(\hat{C}(Y_1, \dots, Y_m))$. Then we can define the ∞-categorical version of the directed category on $X_1, \dots, X_m$ as the dg nerve

$$\hat{\mathcal{C}}(X_1, \dots, X_m) \coloneqq \mathrm{N}_{\mathrm{dg}}(\{Z_1, \dots, Z_m\})$$

of the full dg subcategory of $\mathrm{Perf}(\hat{C}(Y_1, \dots, Y_m))$ spanned by $Z_1, \dots, Z_m$.

We remark that there exists a fully faithful functor $\hat{\mathcal{C}}(X_1, \dots, X_m) \subset \mathcal{D}^{\mathrm{perf}}(\hat{C}(Y_1, \dots, Y_m))$, mapping X_i to the image Z_i.

Remark 2.7. The choice of implementation of the above construction of $\hat{\mathcal{C}}(X_1, \dots, X_m)$ is for illustrative purposes. An equivalent construction in purely ∞-categorical language would be as a full subcategory of the limit of a diagram of stable ∞-categories of the following form:



Here $\mathrm{fib}_{i,i+1}$ is the fiber functor of the i-th and $(i+1)$-th entries. However checking that these two constructions are equivalent involves some explicit computations of the morphism objects in the above limit that we do not wish to unravel here.

2.2 Perverse schobers from collections of spherical objects

Construction 2.8. Let $\mathcal{C}$ be a small k-linear stable ∞-category and $\mathcal{S} = \{S_1, \dots, S_m\} \in \mathcal{C}$ an ordered collection of spherical objects. We define a functor $\mathcal{F}_{\mathcal{S}}\colon \mathrm{Exit}(\mathbf{G}^{\downarrow,m}) \to \mathrm{St}$ as follows:

- We set $\mathcal{F}_{\mathcal{S}}(e_i) = \mathcal{C}$ for all $1 \leq i \leq m$.
- We set $\mathcal{F}_{\mathcal{S}}(v_m) = \mathcal{D}^{\mathrm{perf}}(k)$ and $\mathcal{F}_{\mathcal{S}}(v_m \to e_m) = (\text{-}) \otimes S_m\colon \mathcal{D}^{\mathrm{perf}}(k) \to \mathcal{C}$. Note that $(\text{-}) \otimes S_m$ is a spherical functor by assumption.

- Fix $1 \leq i \leq m-1$. We set $\mathcal{F}_{\mathcal{S}}(v_i)$ to be the following pullback in Cat_∞:

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{perf}}(k) \overrightarrow{\oplus}_{(\text{-})\otimes S_i} \mathcal{C} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \mathrm{ev}_0 \\ \mathcal{D}^{\mathrm{perf}}(k) & \xrightarrow{(\text{-})\otimes S_i} & \mathcal{C} \end{array}$$

  We note that $\mathcal{D}^{\mathrm{perf}}(k) \overrightarrow{\oplus}_{(\text{-})\otimes S_i} \mathcal{C}$ is also called the lax sum along the functor $(\text{-}) \otimes S_i$. We define $\mathcal{F}_{\mathcal{S}}(v_i \to e_{i+1})$ as the composite functor

$$\mathcal{F}_{\mathcal{S}}(v_i \to e_{i+1}) \coloneqq \varrho_1 \colon \mathcal{F}_{\mathcal{S}}(v_i) = \mathcal{D}^{\mathrm{perf}}(k) \overrightarrow{\oplus}_{(\text{-})\otimes S_i} \mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{\mathrm{ev}_1} \mathcal{C}$$

  and $\mathcal{F}_{\mathcal{S}}(v_i \to e_i)$ as the composite functor

$$\mathcal{F}_{\mathcal{S}}(v_i \to e_i) \coloneqq \varrho_2 \colon \mathcal{F}_{\mathcal{S}}(v_i) = \mathcal{D}^{\mathrm{perf}}(k) \overrightarrow{\oplus}_{(\text{-})\otimes S_i} \mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{\mathrm{cof}} \mathcal{C}\,.$$

One can show that the functor $\mathcal{F}_{\mathcal{S}}$ from Construction 2.8 defines a perverse schober, by computing the adjoints of the functor ϱ_1, ϱ_2 and directly checking the conditions. These adjoints are described in [Chr22a, Lemma 3.3, Lemma 3.8].

Lemma 2.9. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{S} = \{S_1, \ldots, S_m\} \subset \mathcal{C}$ be a collection of spherical objects.*

(1) We define the m-th standard object as $\Delta_m \coloneqq \mathrm{ind}^L_{v_m}(k) \in R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$ and the m-th costandard object as $\nabla_m \coloneqq \mathrm{ind}^R_{v_m}(k) \in R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$.

(2) Let $1 \leq i \leq m-1$. We define the i-th standard object as

$$\Delta_i \coloneqq \mathrm{ind}^L_{v_m}((k[-1], 0, 0)) \in R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$$

where $(k[-1], 0, 0) \in \mathcal{D}^{\mathrm{perf}}(k) \overrightarrow{\oplus}_{(\text{-})\otimes S_i} \mathcal{C}$ denotes the essentially unique object which restricts in $\mathcal{D}^{\mathrm{perf}}(k)$ to $k[-1]$ and in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ to $S_i[-1] \to 0$. Similarly, we define the i-th costandard object as

$$\nabla_i \coloneqq \mathrm{ind}^R_{v_m}((k[-1], 0, 0)) \in R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$$

We note that $\mathrm{ev}_{e_1}(\Delta_i) \simeq S_i$ for all $1 \leq i \leq m$, which amounts to the statement that the transport of $\mathcal{F}$ along left turning downwards facing trajectories is trivial.

Proposition 2.10. *Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{S} = \{S_1, \ldots, S_m\} \subset \mathcal{C}$ be a collection of spherical objects.*

(1) The standard objects $\{\Delta_1, \ldots, \Delta_m\} \subset R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$ form a full exceptional collection.

(2) The costandard objects $\{\nabla_1, \ldots, \nabla_m\} \subset R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$ form a full co-exceptional collection, meaning a full exceptional collection after reversing their order.

Proof. We note that $\varrho_1((k[-1], 0, 0)) \simeq 0$ and $\varrho_2((k[-1], 0, 0)) \simeq \mathrm{cof}(S_i[-1] \to 0) \simeq S_i \in \mathcal{C}$. Thus $(k[-1], 0, 0) \in \mathrm{fib}(\mathcal{F}_{\mathcal{S}}(v_i \to e_{i+1})$. It thus follows from Lemma 1.14 that for all $1 \leq i \leq m$, the global sections Δ_i and ∇_i are exceptional. Using the description of the induction functors from [Chr25, Prop. 4.12, Lem. 4.20], one can further show that $\{\Delta_1, \ldots, \Delta_m\}$ are semiorthogonal, as are $\{\nabla_m, \ldots, \nabla_1\}$. We leave to the reader to show that these two exceptional collections are in fact full, meaning that they generate $R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$ under finite limits and colimits. □

Proposition 2.11. *Let $\mathcal{C}$ be a k-linear small stable ∞-category and $\mathcal{S} = \{S_1, \ldots, S_m\}$ an ordered collection of spherical objects in $\mathcal{C}$. Then there exists a fully faithful functor*

$$\hat{\mathcal{C}}(S_1, \ldots, S_m) \subset R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$$

mapping $\hat{S}_i$ to the i-th standard object Δ_i.

Proof. Consider the functor $R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}}) \to \mathcal{F}_{\mathcal{S}}(e_1) = \mathcal{C}$ from the limit cone of $R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}}) = \lim \mathcal{F}_{\mathcal{S}}$. This functor is also denoted by ev_{e_1}.

We choose a dg functor $\mathrm{ev}\colon \mathrm{Perf}(A) \to \mathrm{Perf}(C)$ between dg categories, whose image under $\mathcal{D}^{\mathrm{perf}}(\text{-})\colon \mathrm{dgCat}_k \to \mathrm{St}$ is equivalent to ev_{e_1}. We denote by $\tilde{\Delta}_i \in \mathrm{Perf}(A)$ a choice of object whose image in $R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$ is equivalent to Δ_i and by $\tilde{S}_i \in \mathrm{Perf}(C)$ the image of $\tilde{\Delta}_i$ in $\mathrm{Perf}(C)$.

We finally check that the dg subcategory generated by $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_m$ is quasi-unipotent and that this induces via Lemma 2.6 an equivalence $\mathcal{D}^{\mathrm{perf}}(\hat{C}(S_1 \ldots, S_m)) \simeq R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$, by computing $\mathrm{RHom}_{\mathrm{Perf}(A)}(\tilde{S}_i, \tilde{S}_j) \simeq \mathrm{Mor}_{R\Gamma(\mathbf{G}^{\downarrow,m},\mathcal{F}_{\mathcal{S}})}(\Delta_i, \Delta_j)$. Indeed, we have

$$\begin{aligned}\mathrm{Mor}_{R\Gamma(\mathbf{G}^{\downarrow,m},\mathcal{F}_{\mathcal{S}})}(\Delta_i, \Delta_j) &\simeq \mathrm{Mor}_{\mathcal{F}_{\mathcal{S}}(v_i)}((k[-1], 0, 0), \mathrm{ev}_{v_i}(\Delta_j)) \\ &\simeq \begin{cases} \mathrm{Mor}_{\mathcal{F}_{\mathcal{S}}(v_i)}((k[-1], 0, 0), \mathcal{F}_{\mathcal{S}}(v_i \to e_i)^R(S_j)) & i < j \\ \mathrm{Mor}_{\mathcal{F}_{\mathcal{S}}(v_i)}((k[-1], 0, 0), (k[-1], 0, 0)) & i = j \\ \mathrm{Mor}_{\mathcal{F}_{\mathcal{S}}(v_i)}((k[-1], 0, 0), (0, 0, 0)) & i > j \end{cases} \\ &\simeq \begin{cases} \mathrm{Mor}_{\mathcal{F}_{\mathcal{S}}(e_i)}(S_i, S_j) & i < j \\ k & i = j \\ 0 & i > j \end{cases}\end{aligned}$$

□

2.3 Calabi–Yau completion

The goal of this section is to show that any proper k-linear stable ∞-category with an (always finite) full exceptional collection arises from a directed category on a collection of spherical objects, and hence as the global sections of a perverse schober.

Given a dg category, we denote by $\mathrm{Perf}(C)$ the dg category of compact fibrant-cofibrant dg modules.

We fix a dg algebra A with finite dimensional homology over k. Let $DA = \mathrm{RHom}_k(A, k) \in \mathrm{dgMod}_A^A$. Note that this bimodule corresponds to the Serre functor. We define the square-zero extension dg algebra $A \oplus DA$ with underlying chain complex $A \oplus DA$ and multiplication

$$(a, x) \cdot (a', x') = (aa', a \cdot x' + x \cdot a') \,.$$

Note that $A \oplus DA$ is again finite dimensional and furthermore $D(A \oplus DA) \simeq DA \oplus A \simeq A \oplus DA$ as $(A \oplus DA)$-bimodules. Thus $\mathrm{Perf}(A \oplus DA)$ is a proper, weak right 0-Calabi–Yau dg category.

Replacing DA by $DA[-n]$ in the above construction, we similarly obtain a weak right n-Calabi–Yau dg category $\mathrm{Perf}(A \oplus DA[-n])$. We remark that $A \oplus DA[-n]$ is Koszul-dual to the n-Calabi–Yau completion of Keller [Kel11].

The inclusion $i\colon A \to A \oplus DA[-n]$ induces a functor $F = i_!\colon \mathrm{Perf}(A) \to \mathrm{Perf}(A \oplus DA[-n])$.

Lemma 2.12. *Let $M \in \mathrm{Perf}(A)$ be an exceptional object, i.e. $\mathrm{RHom}_{\mathrm{Perf}(A)}(M, M) \simeq k$. Then $F(M) \in \mathrm{Perf}(A \oplus DA[-n])$ is an n-spherical object, meaning that $\mathrm{RHom}_{A \oplus DA[-n]}(M, M) \simeq k \oplus k[-n]$ and the functor between the derived ∞-categories*

$$(\text{-}) \otimes F(M) \simeq \mathcal{D}^{\mathrm{perf}}((\text{-}) \otimes^L F(M))\colon \mathcal{D}^{\mathrm{perf}}(k) \longrightarrow \mathcal{D}^{\mathrm{perf}}(A \oplus DA[-n])$$

is a spherical functor.

Proof. We may assume that $M \subset A$ is a direct summand in $\mathrm{Perf}(A)$. If this is not the case, we replace A by the dg algebra $B = \mathrm{RHom}_{\mathrm{Perf}(A)}(A \oplus M, A \oplus M)$ and use the observation that the quasi-unital dg functor $A \oplus DA[-n] \subset B \oplus DB[-n]$ is a Morita equivalence.

We now find that

$$\begin{aligned}\mathrm{RHom}_{\mathrm{Perf}(A\oplus DA[-n])}(F(M), F(M)) &\simeq \mathrm{RHom}_{\mathrm{Perf}(A)}(M, M) \oplus \mathrm{RHom}_{\mathrm{Perf}(A)}(M, DM[-n]) \\ &\simeq k \oplus (Dk)[-n] \\ &\subset A \oplus DA[-n] = \mathrm{RHom}_{A\oplus DA[-n]}(A \oplus DA[-n], A \oplus DA[-n])\,.\end{aligned}$$

The right adjoint of the functor $(\text{-}) \otimes^L F(M)$ is given by $\mathrm{RHom}_{\mathcal{D}^{\mathrm{perf}}(A\oplus DA[-n])}(F(M), \text{-})$, which is by the weak right n-Calabi–Yau property equivalent to $\mathrm{RHom}_{\mathcal{D}^{\mathrm{perf}}(A\oplus DA[-n])}(\text{-}, F(M)[n])^*$. Thus, the right adjoint of $\mathrm{RHom}_{\mathcal{D}^{\mathrm{perf}}(A\oplus DA[-n])}(M, \text{-})$ is given by $(\text{-}) \otimes^L F(M)[n]$. Passing to derived ∞-categories and applying [Chr22b, Prop. 4.5], we find that that $(\text{-}) \otimes F(M)$ is a spherical functor. □

Corollary 2.13. *Let $\mathcal{C}$ be a proper k-linear stable ∞-category with a full exceptional collection $\Delta_1, \dots, \Delta_m$. Then there exists a fully faithful functor*

$$\hat{\mathcal{D}}(S_1, \dots, S_m) \subset \mathcal{C}\,,$$

for a proper k-linear stable ∞-category $\mathcal{D}$ and a collection of spherical objects $\mathcal{S} = \{S_1, \dots, S_m\}$ in $\mathcal{D}$, mapping S_i to Δ_i.

Further, by Proposition 2.11, there exists an equivalence of k-linear stable ∞-categories

$$\mathcal{C} \simeq R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})\,,$$

mapping the Δ_i's to the standard objects in $R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$.

Proof. We first note that any algebra object E in the symmetric monoidal ∞-category $\mathcal{D}^{\mathrm{perf}}(k)$ arises from a dg algebra A with finite dimensional homology, see [Lur17, Prop. 7.1.4.6]. Furthermore, $\mathrm{RMod}_E \simeq \mathcal{D}(A)$ as k-linear ∞-categories, see [Chr22a, Prop. 2.19].

Let $\mathrm{End}_{\mathcal{C}}(\bigoplus_{i=1}^m \Delta_i) \in \mathrm{Alg}(\mathcal{D}^{\mathrm{perf}}(k))$ be the endomorphism object in the sense of [Lur17, Section 4.7.1] and A a corresponding dg algebra. We remark that $\mathcal{C}$ is automatically idempotent complete. Thus, by [Lur17, 7.1.2.1], $\mathcal{C} \simeq \mathcal{D}^{\mathrm{perf}}(A)$. We set $\mathcal{D} = \mathcal{D}^{\mathrm{perf}}(A \oplus DA)$. The image of $\Delta_1, \dots, \Delta_m$ under the functor

$$f : \mathcal{C} \simeq \mathcal{D}^{\mathrm{perf}}(A) \longrightarrow \mathcal{D} = \mathcal{D}^{\mathrm{perf}}(A \oplus DA)$$

defines by Lemma 2.12 a collection $\mathcal{S} = \{S_1, \dots, S_m\}$ of spherical objects in $\mathcal{D}$, satisfying that

$$\mathrm{Mor}_{\mathcal{D}}(S_i, S_j) \simeq \begin{cases} \mathrm{Mor}_{\mathcal{C}}(\Delta_i, \Delta_j) & i < j \\ k \oplus k & i = j\,. \end{cases}$$

The functor $\hat{\mathcal{C}}(\Delta_1, \dots, \Delta_m) \to \hat{\mathcal{D}}(S_1, \dots, S_m)$ induced by $\mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories. Since $\hat{\mathcal{C}}(\Delta_1, \dots, \Delta_m)$ is equivalent to a full subcategory of $\mathcal{C}$, we find a fully faithful functor $\hat{\mathcal{D}}(S_1, \dots, S_m) \subset \mathcal{C}$.

Since the image of the functor $\hat{\mathcal{D}}(S_1, \dots, S_m) \to \mathcal{C}$ stably generates $\mathcal{C}$, and the image of the fully faithful functor $\hat{\mathcal{D}}(S_1, \dots, S_m) \to R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}})$ also stably generates, we find by [Lur17, 7.1.2.1] that $R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}_{\mathcal{S}}) \simeq \mathcal{C}$. □

3 Upwards-downwards duality

3.1 Duality for left and right induction

Let $\mathbf{S}$ be the marked surface with spanning graph $\mathbf{G}$. Let v be a vertex of $\mathbf{G}$ with incident edges $e_1, \dots, e_n$. Let further $\mathcal{F}$ be a $\mathbf{G}$-parametrized perverse schober.

We denote by

$$C_v \colon \mathcal{F}(v) \to \mathcal{F}(v)$$

the spherical twist functor of the spherical adjunction

$$\prod_{i=1}^{n} \mathcal{F}(v \to e_i) \colon \mathcal{F}(v) \longleftrightarrow \prod_{i=1}^{n} \mathcal{F}(e_i) : \prod_{i=1}^{n} \mathcal{F}(v \to e_i)^R \,.$$

Let B be the set of external edges of $\mathbf{G}$. Then there is an adjunction arising from evaluation to and induction from the boundary:

$$\prod_{e \in B} \mathrm{ev}_e \colon R\Gamma(\mathbf{G}, \mathcal{F}) \longleftrightarrow \prod_{e \in B} \mathcal{F}(e) : \prod_{e \in B} \mathrm{ind}_e^L \,.$$

This adjunction is spherical, see [Chr25, Cor. 4.7]. We denote its spherical twist by $C_{\mathbf{S}}$.

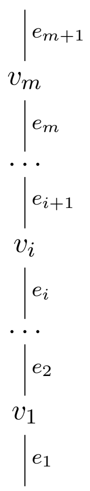
Left and right induction from v to $\mathbf{S}$ are related as follows:

Proposition 3.1 ([Chr25, Prop. 4.23.(2)])**.** *There exists an equivalence of functors*

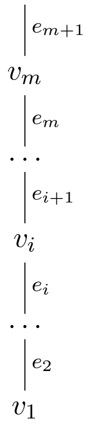
$$C_{\mathbf{S}} \circ \mathrm{ind}_v^L \simeq \mathrm{ind}_v^R \circ C_v \colon \mathcal{F}(v) \longrightarrow R\Gamma(\mathbf{G}, \mathcal{F}) \,.$$

3.2 Perverse schobers on the 2-gon and equivalences of global sections

We denote by $\mathbf{G}^{\updownarrow, m}$ the linear ribbon graph with m vertices and two external edges:



We similarly, denote by $\mathbf{G}^{\uparrow,m}$ the linear ribbon graph:



Let $\mathcal{F}^{\updownarrow}$ be a $\mathbf{G}^{\updownarrow,m}$-parametrized perverse schober. Then replacing $\mathcal{F}(v_m)$ by $\mathrm{fib}(\mathcal{F}(v_m \to e_{m+1})$, and removing $\mathcal{F}(e_{m+1})$, we obtain a $\mathbf{G}^{\downarrow,m}$-parametrized perverse schober, denoted $\mathcal{F}^{\downarrow}$. Similarly, replacing $\mathcal{F}(v_1)$ by $\mathrm{fib}(\mathcal{F}(v_1 \to e_1))$, we obtain the $\mathbf{G}^{\uparrow,m}$-parametrized perverse schober $\mathcal{F}^{\uparrow}$.

Then there are equivalences of stable ∞-categories

$$R\Gamma(\mathbf{G}^{\uparrow,m}, \mathcal{F}^{\uparrow}) \simeq \mathrm{fib}(\mathrm{ev}_{e_1} \colon R\Gamma(\mathbf{G}^{\updownarrow,m}, \mathcal{F}^{\updownarrow}) \to \mathcal{F}^{\updownarrow}(e_1))$$

and

$$R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}^{\downarrow}) \simeq \mathrm{fib}(\mathrm{ev}_{e_{m+1}} \colon R\Gamma(\mathbf{G}^{\updownarrow,m}, \mathcal{F}^{\updownarrow}) \to \mathcal{F}^{\updownarrow}(e_{m+1}))$$

Let $\iota^{\uparrow} \colon R\Gamma(\mathbf{G}^{\uparrow,m}, \mathcal{F}^{\uparrow}) \subset R\Gamma(\mathbf{G}^{\updownarrow,m}, \mathcal{F}^{\updownarrow})$ and $\iota^{\downarrow} \colon R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}^{\downarrow}) \subset R\Gamma(\mathbf{G}^{\updownarrow,m}, \mathcal{F}^{\updownarrow})$ be the arising fully faithful functors.

Proposition 3.2. *The functor $\iota^{\uparrow}$ admits left and right adjoints $\iota^{\uparrow,L}, \iota^{\uparrow,R}$, such that*

$$\iota^{\uparrow,L} \circ \iota^{\downarrow}, \iota^{\uparrow,R} \circ \iota^{\downarrow} \colon R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}^{\downarrow}) \longrightarrow R\Gamma(\mathbf{G}^{\uparrow,m}, \mathcal{F}^{\uparrow})$$

are equivalences of stable ∞-categories.

Proof idea. The functor

$$(\iota^{\uparrow,L}, \iota^{\downarrow,L}) \colon R\Gamma(\mathbf{G}^{\updownarrow,m}, \mathcal{F}^{\updownarrow}) \longrightarrow R\Gamma(\mathbf{G}^{\uparrow,m}, \mathcal{F}^{\uparrow}) \times R\Gamma(\mathbf{G}^{\downarrow,m}, \mathcal{F}^{\downarrow})$$

gives rise to a perverse schober on the 2-spider, arising from the perverse schober on the 2-spider

$$(\mathrm{ev}_{e_1}, \mathrm{ev}_{e_{m+1}}) \colon R\Gamma(\mathbf{G}^{\updownarrow,m}, \mathcal{F}^{\updownarrow}) \longrightarrow \colon \mathcal{F}^{\updownarrow}(e_1) \times \mathcal{F}^{\updownarrow}(e_{m+1})\,.$$

□

Let $\mathcal{C}$ be a k-linear stable ∞-category with a collection of spherical objects $\mathcal{S} = \{S_1, \dots, S_m\}$.

We define the $\mathbf{G}^{\updownarrow,m}$-parametrized perverse schober $\mathcal{F}_{\mathcal{S}}^{\updownarrow}$ as the diagram

$$\begin{array}{c}
\mathcal{C} \\
\big\uparrow{\scriptstyle \varrho_1} \\
\mathcal{D}^{\mathrm{perf}}(k) \overrightarrow{\oplus}_{(\text{-})\otimes S_m} \mathcal{C} \\
\big\downarrow{\scriptstyle \varrho_2} \\
\mathcal{C} \\
\big\uparrow \\
\vdots \\
\big\downarrow \\
\mathcal{C} \\
\big\uparrow{\scriptstyle \varrho_1} \\
\mathcal{D}^{\mathrm{perf}}(k) \overrightarrow{\oplus}_{(\text{-})\otimes S_1} \mathcal{C} \\
\big\downarrow{\scriptstyle \varrho_2} \\
\mathcal{C}
\end{array}$$

We denote $\nabla_i^{\downarrow} \coloneqq \Delta_i$ and $\Delta_i^{\uparrow} = \mathrm{ind}_{v_i}^L((k, S_i, \mathrm{id}_{S_i})) \in R\Gamma(\mathbf{G}^{\uparrow,m}, \mathcal{F}_{\mathcal{S}}^{\uparrow})$.

Proposition 3.3. *(1) There exists an equivalence in* $R\Gamma(\mathbf{G}^{\uparrow,m}, \mathcal{F}_{\mathcal{S}}^{\uparrow})$

$$(\iota^{\uparrow})^L \circ \iota^{\downarrow}(\Delta_i^{\downarrow}) \simeq \nabla_i^{\uparrow}\,.$$

(2) Suppose that S_i *is* n*-spherical. Then*

$$(\iota^{\uparrow})^R \circ \iota^{\downarrow}(\nabla_i^{\downarrow}) \simeq \Delta_i^{\uparrow}[n]\,.$$

Proof. We only prove part (1), the proof of part (2) is similar.

The essential ingredient to prove the above proposition is Proposition 3.1: We have $C_{v_i}((k[-1], 0, 0)) \simeq (k, S_i, \mathrm{id}_{S_i})$. Thus,

$$\begin{aligned}
C_{\mathbf{S}} \circ \iota^{\downarrow}(\Delta_i^{\downarrow}) &\simeq C_{\mathbf{S}} \circ \mathrm{ind}_{v_i}^L((k[-1], 0, 0)) \\
&\simeq \mathrm{ind}_{v_i}^R \circ C_{v_i}((k[-1], 0, 0)) \\
&\simeq \mathrm{ind}_{v_i}^R((k, S_i, \mathrm{id}_{S_i})) \\
&\simeq \iota^{\uparrow}(\nabla_i^{\uparrow})\,.
\end{aligned}$$

We have $(\mathrm{ev}_{e_1}, \mathrm{ev}_{e_{m+1}})(\Delta_i^{\downarrow}) \simeq (S_i, 0)$. Thus:

$$C_{\mathbf{S}} \circ \iota^{\downarrow}(\Delta_i^{\downarrow}) \simeq \mathrm{cof}(\iota^{\downarrow}(\Delta_i^{\downarrow}) \to \mathrm{ind}_{e_1}^L(S_i))\,.$$

We observe that $\mathrm{ev}_{e_1}(C_{\mathbf{S}} \circ \iota^{\downarrow}(\Delta_i^{\downarrow})) \simeq \mathrm{cof}(S_i \to S_i) \simeq 0$, and thus $C_{\mathbf{S}} \circ \iota^{\downarrow}(\Delta_i^{\downarrow}) \in \mathrm{Im}(\iota^{\uparrow})$. By the fully faithfulness of $\iota^{\uparrow}$, we thus obtain

$$C_{\mathbf{S}} \circ \iota^{\downarrow}(\Delta_i^{\downarrow}) \simeq \iota^{\uparrow} \circ (\iota^{\uparrow})^L(C_{\mathbf{S}} \circ \iota^{\downarrow}(\Delta_i^{\downarrow}))\,.$$

Fr any $X \in R\Gamma(\mathbf{G}^{\uparrow,m}, \mathcal{F}_{\mathbf{S}}^{\uparrow})$, we have

$$\begin{aligned}\mathrm{Mor}((\iota^{\uparrow})^L(\mathrm{ind}_{e_1}^L(S_i)), X) &\simeq \mathrm{Mor}_{R\Gamma(\mathbf{G}^{\updownarrow,m},\mathcal{F}_{\mathbf{S}}^{\updownarrow})}(\mathrm{ind}_{e_1}^L(S_i), \iota^{\uparrow}(X)) \\ &\simeq \mathrm{Mor}_{\mathcal{F}(e_1)}(S_i, \underbrace{\mathrm{ev}_{e_1}(\iota^{\uparrow}(X))}_{\simeq 0}) \\ &\simeq 0\end{aligned}$$

This shows (by fully faithfulness of Yoneda) that $(\iota^{\uparrow})^L(\mathrm{ind}_{e_1}^L(S_i)) \simeq 0$.

Therefore,

$$\iota^{\uparrow} \circ (\iota^{\uparrow})^L(C_{\mathbf{S}} \circ \iota^{\downarrow}(\Delta_i^{\downarrow})) \simeq \iota^{\uparrow} \circ (\iota^{\uparrow})^L \circ \iota^{\downarrow}(\Delta_i)\,.$$

In total, we have shown

$$\iota^{\uparrow} \circ (\iota^{\uparrow})^L \circ \iota^{\downarrow}(\Delta_i) \simeq \iota^{\uparrow}(\nabla_i^{\uparrow})$$

from which the assertion follows by the fully faithfulness of $\iota^{\uparrow}$. □

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