∞-categorical group quotients via skew group algebras

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Abstract

We relate group quotients of dg-categories and linear stable ∞-categories. Given a group acting on a dg-algebra, we prove that the skew group dg-algebra is Morita equivalent to the dg-categorical homotopy group quotient. We also treat the cases of group actions on dg-categories, with corresponding skew group dg-categories, and of orbit dg-categories. Finally, we describe a version of the skew group algebra in the setting of ring spectra and relate it with ∞-categorical group quotients.

Contents

1 Introduction

Skew group algebras and skew group dg-categories

The skew group algebra is a classical construction in the representation theory of finite dimensional algebras [RR85] and also appears in the construction of non-commutative resolutions of singularities [KV00, VdB22]. As input for its construction serves an algebra A over a field k and a group G acting on A by automorphisms. The skew group algebra AG has as underlying vector space the coproduct $A^{\amalg G}$, with multiplication determined on additive generators by $(g_1, a_1) \cdot (g_2, a_2) = (g_1 g_2, a_1 \cdot (g_1.a_2))$. An analogue of this construction is also known in the case that A is a dg-algebra, dg-category or even an A_∞-category, see for instance [Meu20, OZ22, AP24].

The representation theory of the skew group algebra AG is typically better behaved than that of the algebra of fixed points A^G, which is why AG has become a standard construction. Indeed, under some finiteness assumptions, the module category $\mathrm{mod}(AG)$ is equivalent to the category $\mathrm{mod}(A)^G$ of G-equivariant A-modules, see [Dem11].

By passing to the setting of dg-categories, we can clarify the universal property of the skew group algebra:

Proposition 1.1 (Proposition 3.5 and Remark 3.6)**.** *Fix a base field k. Let A be a dg-category with a strict action by a group G. We consider the G-action as a functor*

$$\rho\colon BG \to \mathrm{dgCat}_k\,,$$

where BG is the classifying space of G. The homotopy colimit[1] *of ρ is Morita equivalent to the skew group dg-category, denoted $A * G$.*

If A has a single object, we can identify it with a dg-algebra and in this case $A * G = AG$ describes a dg-version of the skew group algebra.

The above homotopy colimit is also called the homotopy group quotient of A by G. The homotopy limit of ρ can be seen as the dg-category of G-equivariant perfect dg-modules. The advantage of using homotopy colimits instead of homotopy limits is that it allows to avoid any finiteness assumptions. Note also the novelty that we impose no assumptions on the characteristic of the ground field k.

In the case $G = \mathbb{Z}$, we will also treat non-strict actions and show that the orbit dg-category, recently constructed for non-strict $\mathbb{Z}$-actions in [FKQ24], describes the homotopy colimit in Proposition 4.5.

To prove Proposition 1.1, we construct a strict cocone under a diagram equivalent to $\rho\colon BG \to \mathrm{dgCat}_k$, whose tip is Morita equivalent to AG. Passing to derived ∞-categories, we obtain an induced functor from the ∞-categorical colimit $\mathcal{D}(A)_G$ to the derived ∞-category of AG. We use the ∞-categorical colimit $\mathcal{D}(A)_G$ to describe the homotopy colimit, via the equivalence of dg-categories up to Morita equivalence with compactly generated k-linear stable ∞-categories. This functor $\mathcal{D}(A)_G \to \mathcal{D}(AG)$ is an equivalence, which is shown by comparing the morphism objects of generators.

We also give a separate proof of Proposition 1.1 under some cofibrancy assumptions, by exhibiting the skew-group algebra as the strict dg-categorical colimit of a cofibrant diagram, see Proposition 3.10.

The results of this note will not surprise experts, but instead fill a gap in the literature and attempt to unify different perspectives on group quotients. Exhibiting ∞-categorical universal properties of the dg-categorical constructions makes these accessible to powerful ∞-categorical arguments. For instance, an immediate consequence of the above results is that the passage to skew group dg-algebras and skew group dg-categories commutes with ∞-categorical colimits in $\mathrm{LinCat}_{\mathrm{Mod}_k}$, as colimits commute with colimits. Reversing the perspective, the results of this note also serve to obtain concrete models for the abstract ∞-categorical constructions. We illustrate this by describing periodic derived ∞-categories and periodic topological Fukaya categories of surfaces in terms of orbit categories in Section 4.3.

Group actions on linear stable ∞-categories

Let G be a group. The classifying space BG is the category with a single object $*$ with endomorphisms G. We work over a base $\mathbb{E}_\infty$-ring spectrum R and denote Mod_R denote the symmetric monoidal stable ∞-category of R-module spectra. We denote by

$$\mathrm{LinCat}_{\mathrm{Mod}_R} = \mathrm{LMod}_{\mathrm{Mod}_R}(\mathcal{P}r^L)$$

the ∞-category of Mod_R-linear presentable ∞-categories (these are automatically stable).

The standard way to define an action of G on a Mod_R-linear ∞-category $\mathcal{C}$ is as a functor $BG \to \mathrm{LinCat}_{\mathrm{Mod}_R}$, mapping $*$ to $\mathcal{C}$. Central to our treatment of group actions will be the following equivalent way to such G-actions, see for instance [CCRY22, BMCSY23]: forming the coproduct of Mod_R over G yields a monoidal ∞-category $\mathrm{Mod}_R^{\amalg G} \in \mathrm{LinCat}_{\mathrm{Mod}_R}$, called the categorical group algebra. A G-action on $\mathcal{C}$ can equivalently be expressed as a left $\mathrm{Mod}_R^{\amalg G}$-action on $\mathcal{C}$.

[1] With respect to the Morita model structure on the category dgCat_k of dg-categories.

This perspective allows to apply the powerful and well developed framework of ∞-categorical algebra objects and modules of [Lur17] to the study of group actions and group quotients. For instance, the ∞-categorical colimit over BG amounts in terms of left actions to tensoring with the Mod_R-$\mathrm{Mod}_R^{\amalg G}$-bimodule Mod_R. Using this perspective, desired basic properties of group quotients readily follow.

Skew group ring spectra

In Section 5, we generalize the construction of the skew group algebra to the setting of a group G acting on an R-linear ring spectrum A. The skew group algebra AG is then an R-linear ring spectrum with underlying spectrum the coproduct $A^{\amalg G}$, and the multiplication generalizing the above. The actual construction of the ring spectrum AG is based on universal constructions, see Remark 5.4 for a summary. Proposition 1.1 then generalizes as follows:

Theorem 1.2. *Let R be the base $\mathbb{E}_\infty$-ring spectrum and $A \in \mathrm{Alg}(\mathrm{Mod}_R)$ be an R-linear ring spectrum. Let $\rho\colon BG \to \mathrm{Alg}(\mathrm{Mod}_R)$ be an action of a group G on A. The colimit of the functor between ∞-categories*

$$BG \xrightarrow{\rho} \mathrm{Alg}(\mathrm{Mod}_R) \xrightarrow{\mathrm{RMod}_{(-)}} \mathrm{LinCat}_{\mathrm{Mod}_R} \tag{1}$$

is equivalent to RMod_{AG}.

We remark that the limit of the functor (1) is equivalent to its colimit, since we are in the setting of presentable ∞-categories, and hence to RMod_{AG}, see Lemma 2.7.

Comparison with previous results

Classical skew group algebras

Given a category C with an action by a group G, we denote by C^G the category of G-equivariant objects. Note that C^G describes the homotopy limit of a functor $BG \to \mathrm{Cat}$, mapping the basepoint of BG to C, see Remark 3.8. Given a (not necessarily finite dimensional) algebra A over a commutative ring k, we denote by $\mathrm{mod}(A)$ the abelian category of finite dimensional right A-modules and by $\mathrm{Mod}(A)$ the (non-small) abelian category of right A-modules.

Proposition 1.3 ([Dem11, Prop. 2.48])**.** *Let k be a field and let G be finite group whose order does not divide the characteristic of k. Suppose that G acts on a finite dimensional k-algebra A. Then there exists an equivalence of categories*

$$\mathrm{mod}(A)^G \simeq \mathrm{mod}(AG)\,.$$

The above equivalence of categories extends to the (bounded) derived 1-categories, using that $\mathcal{D}^b(\mathrm{mod}(A)^G) \simeq \mathcal{D}^b(\mathrm{mod}(A))^G$, see [Ela14, Thm. 7.1] or [Che15,Bal11]. Conversely, the equivalence of module categories from Proposition 1.3 can be recovered from the equivalence of derived categories by restricting to the hearts. Theorem 1.2 and Lemma 2.7 thus allow us to generalize Proposition 1.3:

Corollary 1.4. *Let k be a commutative ring and G a group acting on a k-algebra A. There exists an equivalence of derived ∞-categories*

$$\mathcal{D}(A * G) \simeq \mathcal{D}(A)^G \coloneqq \lim_{BG} \mathcal{D}(A)\,.$$

Passing to homotopy 1-categories, this equivalence restricts to an equivalence of abelian subcategories $\mathrm{Mod}(A * G) \simeq \mathrm{Mod}(A)^G$.

Note that if furthermore G is finite and A finite dimensional, then this equivalence restricts to the finite dimensional module categories: $\mathrm{mod}(A * G) \simeq \mathrm{mod}(A)^G$.

The proof is given in Section 3.1.

Ring spectra

Let R be the base $\mathbb{E}_\infty$-ring spectrum. Given a group G, the skew group algebra RG of the trivial G-action on R recovers the R-linear group algebra of G. In this case, Theorem 1.2 recovers the well known equivalence of ∞-categories:

$$\mathrm{RMod}_{RG} \simeq \mathrm{Fun}(BG, \mathrm{Mod}_R) \simeq \mathrm{colim}_{BG}\, \mathrm{Mod}_R\ .$$

Previous results in the literature also concern an analogue of Theorem 1.2 in the case that $A = R$ is the base ring spectrum and G is a monoid in the ∞-category of spaces. The restriction $A = R$ corresponds to the assertion that the functor $BG \to \mathrm{LinCat}_{\mathrm{Mod}_R}$ is pointed. We refer to [Dou05, Prop. 3.13] or [HM23, Thm. 0.0.7] in case of $R = \mathbb{S}$ the sphere spectrum, and [CCRY22, Thm. 7.13] for general R. The analogue of the skew group algebra in this setting is also called the Thom spectrum.

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Higher categorical preliminaries

We generally follow the notation and conventions of [Lur09, Lur17].

- We consider dg-categories over a fixed base field k. The derived ∞-category $\mathcal{D}(\mathcal{B})$ of a dg-category $\mathcal{B}$ is defined as the Ind-completion of the dg-nerve $\mathrm{Ind}\, N_{\mathrm{dg}}(\mathrm{Perf}(\mathcal{B}))$ of the dg-category of cofibrant compact right dg-B-modules. The passage to the derived ∞-category defines a colimit preserving and reflecting functor between ∞-categories

  $$\mathcal{D}(\text{-})\colon N(\mathrm{dgCat}_k)[M^{-1}] \to \mathrm{LinCat}_{\mathrm{Mod}_k}\,,$$

  with M the collection of Morita equivalences.
- R will usually denote a base $\mathbb{E}_\infty$-ring spectrum. Given a Mod_R-linear ∞-category $\mathcal{D}$ and $X, Y \in \mathcal{D}$, we write $\mathrm{Mor}_{\mathcal{D}}(X, Y) \in \mathrm{Mod}_R$ for the Mod_R-linear morphism object.
- Given a regular cardinal κ, we denote by $\mathrm{LinCat}^\kappa_{\mathrm{Mod}_R} = \mathrm{Mod}_{\mathrm{Mod}_R}(\mathcal{P}r^L_\kappa)$ the presentable ∞-category of Mod_R-modules in the presentable ∞-category $\mathcal{P}r^L_\kappa$ of κ-compactly generated presentable ∞-categories, as in [Lur17, Notation 5.3.2.8]. Its presentability is the advantage of $\mathrm{LinCat}^\kappa_{\mathrm{Mod}_R}$ over $\mathrm{LinCat}_{\mathrm{Mod}_R}$.

  Given a Mod_R-linear monoidal ∞-category $\mathcal{C} \in \mathrm{Alg}(\mathrm{LinCat}_{\mathrm{Mod}_R})$, we choose a sufficiently large regular cardinal κ as in [Lur17, Lem. 5.3.2.12], for which in particular $\mathcal{C}$ is κ-compactly generated. We also denote

  $$\mathrm{LinCat}^\kappa_{\mathcal{C}} = \mathrm{LMod}_{\mathcal{C}}(\mathrm{LinCat}^\kappa_{\mathrm{Mod}_R})$$

  for the presentable ∞-category of κ-compactly generated $\mathcal{C}$-linear categories.
- Given a presentable monoidal ∞-category $\mathcal{C}$, in Sections 4.8.3-4.8.5 of [Lur17], Lurie describes a fully faithful (see [Lur17, Thm. 4.8.5.5]) functor[2]

  $$\Theta_*\colon \mathrm{Alg}(\mathcal{C}) \longrightarrow (\mathrm{LinCat}_{\mathcal{C}})_{\mathcal{C}/}\,,$$

[2]More precisely, the functor Θ_* considered here is a restriction the functor Θ_* of [Lur17]. We leave the choice of $\mathcal{C}$ in the notation for Θ_* implicit.

mapping an algebra object $A \in C$ to its ∞-category $\mathrm{RMod}_A(\mathcal{C})$ of right modules, together with the functor $- \otimes A\colon \mathcal{C} \to \mathrm{RMod}_A(\mathcal{C})$ mapping the monoidal unit to A. The image of the functor Θ_* is concretely characterized in [Lur17, Prop. 4.8.5.8]. The functor Θ is defined as the composite of Θ_* with the forgetful functor

$$(\mathrm{LinCat}_{\mathrm{Mod}_R})_{\mathrm{Mod}_R/} \to \mathrm{LinCat}_{\mathrm{Mod}_R} .$$

2 ∞-categorical group actions

For the entire section, we fix a group G (which is not required to be finite).

2.1 The categorical group algebra

Given a set (or a space) X, we denote by $\mathrm{Mod}_R^{\amalg X}$ the colimit over X of the constant diagram in $\mathrm{LinCat}_{\mathrm{Mod}_R}$ with value Mod_R. Note that $\mathrm{Mod}_R^{\amalg X} \simeq \mathrm{Mod}_R^{\times X}$ also describes the limit over X, which follows from the equivalence $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{\mathrm{op}}$.

There is an essentially unique colimit preserving functor

$$\mathrm{Mod}_R^{\amalg(\text{-})}\colon \mathcal{S} \to \mathrm{LinCat}_{\mathrm{Mod}_R} = \mathrm{Mod}_{\mathrm{Mod}_R}(\mathcal{P}r^L)\,,$$

determined by mapping $*$ to Mod_R. Furthermore, since $\mathcal{S}$ is the unit object in the symmetric monoidal ∞-category $\mathcal{P}r^L$, the functor $\mathrm{Mod}_R^{\amalg}$ lifts essentially uniquely to a symmetric monoidal functor $(\mathrm{Mod}_R^{\amalg(\text{-})})^{\otimes}\colon \mathcal{S}^{\otimes} \to \mathrm{LinCat}_{\mathrm{Mod}_R}^{\otimes}$, see [Lur17, Prop. 3.2.1.8].

Construction 2.1. The group G gives rise to an Assoc-algebra object in $\mathrm{Set}^{\otimes}$, i.e. a morphism of ∞-operads $\mathrm{Assoc}^{\otimes} \to \mathrm{Set}^{\otimes}$ over $N(\mathrm{Fin}_*)$, see also [Lur17, Def. 4.1.1.3] for the definition of $\mathrm{Assoc}^{\otimes}$, where the symmetric monoidal structure of $\mathrm{Set}^{\otimes}$ is the Cartesian symmetric monoidal structure.

Composing with the inclusion of symmetric monoidal ∞-categories $\mathrm{Set}^{\otimes} \subset \mathcal{S}^{\otimes}$ and the symmetric monoidal functor $(\mathrm{Mod}_R^{\amalg(\text{-})})^{\otimes}$, we obtain an Assoc-algebra object $\mathrm{Mod}_R^{\amalg G} \in \mathrm{Alg}_{\mathrm{Assoc}}(\mathrm{LinCat}_{\mathrm{Mod}_R})$, meaning a Mod_R-linear monoidal ∞-category.

We note that any group homomorphism $G \to G'$ induces a monoidal functor $\mathrm{Mod}_R^{\amalg G} \to \mathrm{Mod}_R^{\amalg G'}$.

Definition 2.2. The monoidal ∞-category $\mathrm{Mod}_R^{\amalg G}$ from Construction 2.1 is called the categorical group algebra of G.

Given $g \in G$, we denote the object of $\mathrm{Mod}_R^{\amalg G}$ lying in the g-th component with value R by R_g. The monoidal unit of $\mathrm{Mod}_R^{\amalg G}$ is given by R_e.

Remark 2.3. The unique group homomorphism $\psi\colon G \to \{e\}$ gives by Construction 2.1 rise to a monoidal functor $\psi\colon \mathrm{Mod}_R^{\amalg G} \to \mathrm{Mod}_R$. Explicitly, ψ maps a G-tuple $(A_g)_{g\in G}$ to $\coprod_{g\in G} A_g$.

Lemma 2.4.

(1) The monoidal ∞-category $\mathrm{Mod}_R^{\amalg G}$ *is locally rigid in the sense of* [Lur18, Def. D.7.4.1].

(2) If $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathrm{Mod}_R^{\amalg G}$-linear functor whose right adjoint G preserves colimits, then G is also $\mathrm{Mod}_R^{\amalg G}$-linear.

Proof. We begin with showing the local rigidity. Conditions (1) and (2) of Definition D.7.4.1 are clear. The unit object of $\mathrm{Mod}_R^{\amalg G}$ is given by the object R_e, with e the unit of G, and clearly compact, giving condition (3).

The subset of left and right dualizable objects is closed under finite limits and colimits. For condition (4), it thus suffices to check that each object R_g with $g \in G$ admits a left and right dual. Since R_g is invertible with inverse $R_{g^{-1}}$, this is clear.

Part (2) of the Lemma follows from part (1) by [Lur18, Rem. D.7.4.4]. The broken reference at the end of loc. cit. may refer to the fact that the R-linear structure on G in question is induced by the R-linear structure of F via [Lur17, Cor. 7.3.2.12], so that the existence of the R-linear structure in question is a property. □

Example 2.5. We can consider the monoidal functor $\psi\colon \mathrm{Mod}_R^{\amalg G} \to \mathrm{Mod}_R$ as a $\mathrm{Mod}_R^{\amalg G}$-linear functor. Its right adjoint $\psi^R\colon \mathrm{Mod}_R \to \mathrm{Mod}_R^{\amalg G}$, G-componentwise given by the identity functor, inherits by Lemma 2.4 the structure of a $\mathrm{Mod}_R^{\amalg G}$-linear functor.

The composite $\psi^R \circ \psi\colon \mathrm{Mod}_R^{\amalg G} \to \mathrm{Mod}_R^{\amalg G}$ is equivalent to $\coprod_{g\in G} R_g \otimes (\text{-})$ and inherits the structure of a monad.

If G is a finite group, then the right adjoint ψ^{RR} of ψ^R agrees with ψ. The inherited $\mathrm{Mod}_R^{\amalg G}$-linear structure of ψ^{RR} also coincides with that of ψ. This follows from the fact that $\mathrm{Mod}_R^{\amalg G}$ is the free $\mathrm{Mod}_R^{\amalg G}$-linear ∞-category generated by a point, so that both linear functors are determined by the value of the monoidal unit R_e.

2.2 Group actions and group quotients

We denote by BG the classifying space of G, meaning the 1-category with a unique object $*$ and endomorphisms G. We will not distinguish between BG and its nerve $N(BG) \in \mathrm{Set}_\Delta$ in notation.

A G-action on an object $\mathcal{C}$ in an ∞-category $\mathscr{C}$ is understood to be a functor $BG \to \mathscr{C}$, mapping $*$ to $\mathcal{C}$.

Definition 2.6. Let $\mathscr{C}$ be an ∞-category that admits limits and colimits. Given G-action $\rho\colon BG \to \mathscr{C}$ on $\mathcal{C} \in \mathscr{C}$, we call

- the limit $\mathcal{C}^G := \lim(\rho)$ the fixed points of the G-action on $\mathcal{C}$.
- the colimit $\mathcal{C}_G := \mathrm{colim}(\rho)$ the group quotient of $\mathcal{C}$ by G.

Lemma 2.7. *Consider a functor $\rho_{\mathcal{C}}\colon BG \to \mathrm{LinCat}_{\mathrm{Mod}_R}$ describing a G-action on a Mod_R-linear ∞-category $\mathcal{C}$. Its colimits $\mathcal{C}_G$ is equivalent to its limit $\mathcal{C}^G$ in $\mathrm{LinCat}_{\mathrm{Mod}_R}$.*

Proof. The conservative functor $\mathrm{LMod}_{\mathrm{Mod}_R^{\amalg G}}(\mathcal{P}r^L) \to \mathcal{P}r^L$ reflects limits and colimits and preserves both by [Lur17, Cor. 3.4.3.6, 3.4.4.6]. The right adjoint diagram of the colimit diagram of $\rho_{\mathcal{C}}$ again lies in $\mathrm{LMod}_{\mathrm{Mod}_R^{\amalg G}}(\mathcal{P}r^L)$ by Lemma 2.4. It is a limit diagram, since its underlying diagram in $\mathcal{P}r^L$ is, as follows from the duality $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{\mathrm{op}}$ and the fact that the functors $\mathcal{P}r^R, \mathcal{P}r^L \to \mathrm{Cat}_\infty$ preserve and reflect limits. Note that $BG^{\mathrm{op}} \simeq BG$, since BG is a space, hence the right adjoint diagram of $\rho_{\mathcal{C}}$ is equivalent to $\rho_{\mathcal{C}}$. □

We choose a sufficiently large regular cardinal κ. G-actions on Mod_R-linear ∞-categories can be equivalently expressed as follows:

Proposition 2.8 ([CCRY22,BMCSY23])**.** *There exists a canonical equivalence of ∞-categories*

$$\mathrm{Fun}(BG, \mathrm{LinCat}^\kappa_{\mathrm{Mod}_R}) \simeq \mathrm{LinCat}^\kappa_{\mathrm{Mod}_R^{\amalg G}} .$$

Proof. This is [CCRY22, Lem. 4.49] applied in the case $\mathscr{C} = \mathrm{LinCat}^\kappa_{\mathrm{Mod}_R}$ and $A = G$. □

Remark 2.9. The relation between a G-action $\rho\colon BG \to \mathrm{LinCat}^\kappa_{\mathrm{Mod}_R}$ and the $\mathrm{Mod}_R^{\amalg G}$-linear structure on a given Mod_R-linear ∞-category $\mathcal{C}$ can be explicitly described as follows: given $g \in G$, its action $\rho(g)\colon \mathcal{C} \simeq \mathcal{C}$ is equivalent to the functor $R_g \otimes \text{-}\colon \mathcal{C} \simeq \mathcal{C}$.

Lemma 2.10 ([CCRY22, Lem. 4.51])**.** *Consider a group action $BG \to \mathrm{LinCat}_{\mathrm{Mod}_R}$ on $\mathcal{C}$ and the corresponding $\mathrm{Mod}_R^{\amalg G}$-linear structure of $\mathcal{C}$ of Proposition 2.8.*

(1) The group quotient of $\mathcal{C}$ is equivalent to the relative tensor product:

$$\mathcal{C}_G \simeq \mathrm{Mod}_R \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathcal{C} \in \mathrm{LinCat}_{\mathrm{Mod}_R} .$$

(2) The functor $\mathcal{C} \to \mathcal{C}_G$ from the colimit cone is equivalent to $\psi \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathcal{C}$, with ψ as in Remark 2.3.

Proof. Consider the morphisms of spaces $i\colon * \to BG$ and $\pi\colon BG \to *$. The morphism π induces under the functor $\mathrm{Fun}(\text{-}, \mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R})\colon \mathcal{S} \to \mathrm{Mod}_{\mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R}}$ the colimit functor $\mathrm{Fun}(BG, \mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R}) \to \mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R}$. By [CCRY22, Lem. 4.51], this colimit functor is given by the tensor product functor $\mathrm{Mod}_R \otimes_{\mathrm{Mod}_R^{\amalg} G}(\text{-})$. Part (1) now follows from the fact that the inclusion $\mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R} \subset \mathrm{LinCat}_{\mathrm{Mod}_R}$ preserves colimits [Lur17, Lem. 5.3.2.9].

For part (2), we note that by [CCRY22, Lem. 4.51], the natural equivalence colim $\simeq (\text{-}) \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathrm{Mod}_R\colon \mathrm{Fun}(BG, \mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R}) \to \mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R}$ lies under $\mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R}$. The functor $\mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R} \to \mathrm{Fun}(BG, \mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R})$ is given by $i_!$, and passing to the right adjoint induces the natural transformation

$$i^* \simeq \mathrm{colim}\, i_! i^* \xRightarrow{\text{counit}} \mathrm{colim}\colon \mathrm{Fun}(BG, \mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R}) \to \mathrm{LinCat}^{\kappa}_{\mathrm{Mod}_R}\,,$$

which evaluates at $\mathcal{C}$ to functor $\mathcal{C} \to \mathcal{C}_G$ from the colimit cone. The desired description of this natural transformation follows from the fact that the counit

$$\text{counit}\colon i_! i^* \simeq (\text{-}) \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathrm{Mod}_R^{\amalg G} \otimes_{\mathrm{Mod}_R} \mathrm{Mod}_R^{\amalg G} \Longrightarrow (\text{-}) \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathrm{Mod}_R \otimes_{\mathrm{Mod}_R} \mathrm{Mod}_R^{\amalg G} \simeq \mathrm{id}$$

is given by $(\text{-}) \otimes_{\mathrm{Mod}_R^{\amalg G}} \psi \otimes_{\mathrm{Mod}_R} \mathrm{Mod}_R^{\amalg G}$. ☐

Lemma 2.11. *Consider a group action $\rho_{\mathcal{C}}\colon BG \to \mathrm{LinCat}_{\mathrm{Mod}_R}$ on $\mathcal{C}$ and the corresponding $\mathrm{Mod}_R^{\amalg G}$-linear structure.*

(1) The functor

$$\psi^R \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathcal{C}\colon \mathcal{C}_G \simeq \mathrm{Mod}_R \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathcal{C} \longrightarrow \mathrm{Mod}_R^{\amalg G} \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathcal{C} \simeq \mathcal{C} \tag{2}$$

is monadic.

(2) The endofunctor of $\mathcal{C}$ underlying the adjunction monad $\psi^R\psi \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathcal{C}$ of the adjunction $\psi \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathcal{C} \dashv \psi^R \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathcal{C}$ is given by $\coprod_{g\in G} \rho_{\mathcal{C}}(g)$.

(3) If G is finite, then the functor (2) is also left adjoint to $\mathcal{C} \otimes_{\mathrm{Mod}_R^{\amalg G}} \psi\colon \mathcal{C} \to \mathcal{C}_G$.

Proof. We begin with showing (1). By Lemma 2.7, $\mathcal{C}_G$ is equivalent to the limit of $\rho_{\mathcal{C}}$, and hence to the ∞-category of coCartesian sections of the Grothendieck construction of $\rho_{\mathcal{C}}$, see [Lur24, Prop. 05RX]. Under this equivalence, the functor

$$\mathcal{C} \otimes_{\mathrm{Mod}_R^{\amalg G}} \psi^R\colon \mathcal{C}_G \simeq \mathrm{colim}(\rho_{\mathcal{C}}) \longrightarrow \mathcal{C}$$

evaluates a coCartesian section at $* \in BG$. Thus, the functor is clearly conservative and hence monadic by [Lur17, Thm. 4.7.3.5].

For (2), we note the equivalence

$$\psi^R\psi \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathcal{C} \simeq \coprod_{g\in G} (R_g \otimes (\text{-})) \otimes_{\mathrm{Mod}_R^{\amalg G}} \mathcal{C} \simeq \coprod_{g\in G} \rho_{\mathcal{C}}(g)\,.$$

For (3), we note that by Example 2.5, if G is finite, the $\mathrm{Mod}_R^{\amalg G}$-linear functor ψ^R is also left adjoint to ψ, which induces the desired adjunction. ☐

Remark 2.12. We expect that Statement (3) of Lemma 2.11 also holds in the more general case where G is a monoid in sets, such as $G = \mathbb{N}$, or even a monoid of spaces. However, the proof of the statement uses the $\mathrm{Mod}_R^{\amalg G}$-linearity of ψ^R, which fails already for $G = \mathbb{N}$.

3 Skew group dg-categories

In this section, we consider a dg-category A with a strict action by a group G. In Section 3.1, we relate the skew group dg-category with the ∞-categorical group quotient. In Section 3.2, we describe under some conditions the skew group dg-category as the colimit of a cofibrant diagram.

3.1 Skew group dg-categories as ∞-categorical group quotients

We fix a group G with a strict action on a dg-category A. We can equivalently consider the action as a functor $\rho_A\colon BG \to \mathrm{dgCat}_k$ mapping $*$ to A. We define the skew group dg-category $A * G$ as follows, analogous to the definition of the skew group A_∞-category in [OZ22, Def. 5.6], [AP24, Def. 2.11].

- The set of objects of $A * G$ is given by the set of objects of A. We will however write $\tilde{X} \in A * G$ for the object corresponding to $X \in A$.
- Given $\tilde{X}, \tilde{Y} \in A * G$, the morphism chain complex is given by

$$\mathrm{Map}_{A*G}(\tilde{X}, \tilde{Y}) = \coprod_{g\in G} \mathrm{Map}_A(g.X, Y)\,.$$

  Given a morphism $a\colon g.X \to Y$ in A, we write $(g, a)\colon \tilde{X} \to \tilde{Y}$ for the corresponding morphism in $A * G$. Every morphism in $A * G$ can be uniquely written as a sum $\sum_{g\in G}(g, a_g)$, with finitely many non-zero summands.
- the composition map

$$(\text{-}) \circ (\text{-})\colon\ \mathrm{Map}_{A*G}(\tilde{X}, \tilde{Y}) \times \mathrm{Map}_{A*G}(\tilde{Y}, \tilde{Z}) \longrightarrow \mathrm{Map}_{A*G}(\tilde{X}, \tilde{Z})$$

  is defined on generators by $(g_2, b) \circ (g_1, a) \coloneqq (g_2g_1, b \circ (g_2.a))$.

There is an apparent dg-functor $F_A\colon A \to A * G$, given by the assignments $X \mapsto \tilde{X}$ on objects and $a \mapsto (e, a)$ on morphisms, where $e \in G$ is the unit element.

We can define a G-action $\rho_{A*G}\colon BG \to \mathrm{dgCat}_k$ on $A*G$ by letting $h \in G$ act as $h.\tilde{X} = \widetilde{h.X}$ and $h.(g, a) = (hgh^{-1}, h.a)$. Indeed, we find:

$$\begin{aligned} h.\left((g_2, b) \circ (g_1, a)\right) &= (hg_2g_1h^{-1}, h.(b \circ (g_2.a))) \\ &= (hg_2h^{-1}, h.b) \circ (hg_1h^{-1}, h.a) = h.(g_2, b) \circ h.(g_1, a) \end{aligned}$$

We observe the following:

Lemma 3.1. *The functor $F_A\colon A \to A*G$ extends to a morphism $\rho_A \to \rho_{A*G}$ in $\mathrm{Fun}(BG, \mathrm{dgCat}_k)$.*

Definition 3.2. We choose for every G-orbit $[X]$ of objects in A an arbitrary representative $X \in A$. The dg-category $(A * G)^{\mathrm{red}}$ is defined as the full dg-subcategory of $A * G$ on the objects of the form $\tilde{X}$, with X a chosen representative.

We note that the inclusion $(A * G)^{\mathrm{red}} \subset A * G$ is an equivalence of dg-categories.

Lemma 3.3. *Suppose that the action of G is free on the set of objects of A. Then the G-action on $(A * G)^{\mathrm{red}}$ induced by the equivalence of dg-categories $(A * G)^{\mathrm{red}} \simeq A * G$ is trivial.*

Proof. We unraveling the induced G-action on $(A * G)^{\mathrm{red}}$. Fix $h \in G$. The action of h is given by the composite

$$(A * G)^{\mathrm{red}} \simeq A * G \xrightarrow{\rho_{A*G}(h)} \mathcal{C} * G \simeq (A * G)^{\mathrm{red}}\,.$$

This action clearly acts as the identity on objects. Given a morphism $(g, a)\colon \tilde{X} \to \tilde{Y}$ in $(\mathcal{C} * G)^{\mathrm{red}}$, the action of h is given by the composite

$$\tilde{X} \xrightarrow{(h^{-1},\mathrm{id}_X)} \widetilde{h.X} \xrightarrow{(hgh^{-1},h.a)} \widetilde{h.Y} \xrightarrow{(h,\mathrm{id}_Y)} \tilde{Y}$$

which is by definition again given by (g, a). $\square$

We can always arrange the G-action on A to be free on the set of objects:

Lemma 3.4. *The dg-category a with the group action by G is equivalent in $\mathrm{Fun}(BG, \mathrm{dgCat}_k)$ to a dg-category A' with a G-action that is free on its set of objects.*

Proof. The set $\mathrm{ob}(A)$ of objects of A splits into the orbits of the G-action $\mathrm{ob}(A) \coprod_{[x] \in \mathrm{ob}(A)/G} G.x$. We choose a set of representatives $R \subset \mathrm{ob}(A)$ of the orbits $\mathrm{ob}(A)/G$. We define the set of objects $\mathrm{ob}(A')$ of A' to be pairs (x, g) with $x \in R$ and $g \in G$. The morphism complexes are defined as $\mathrm{Map}_{A'}((x, g), (y, h)) = \mathrm{Map}_A(g.x, h.y)$ with composition as in A. The apparent dg-functor $\pi \colon A' \to A$, given on objects by $(x, g) \to g.x$ is fully faithful and essentially surjective, and hence an equivalence of dg-categories. The G-action on A induces a G-action on A', given on objects by $h.(x, g) = (x, hg)$ and on morphisms complexes as for A. The equivalence of dg-categories π is indeed G-equivariant. □

Let A' be as in Lemma 3.4. Then $(A' * G)^{\mathrm{red}}$ is by Lemma 3.3 the tip of a cocone under the diagram $BG \to \mathrm{dgCat}_k$ describing the G-action on A'. Passing to derived ∞-categories, we see that

$$\mathcal{D}((A' * G)^{\mathrm{red}}) \simeq \mathcal{D}(A' * G) \simeq \mathcal{D}(A * G)$$

defines the tip of a cocone under the diagram

$$\rho_{\mathcal{D}(A)} \colon BG \to \mathrm{LinCat}_{\mathrm{Mod}_k}$$

describing the G-action on $\mathcal{D}(A)$. We denote the colimit of the functor $\rho_{\mathcal{D}(A)}$ by $\mathcal{D}(A)_G$. By its universal property, there is an induced functor

$$\zeta \colon \mathcal{D}(A)_G \longrightarrow \mathcal{D}(A * G)$$

Proposition 3.5. *The above functor ζ is a an equivalence of k-linear ∞-categories.*

Remark 3.6. The ∞-category underlying the model category dgCat_k with the Morita model structure is equivalent to the ∞-category $\mathrm{LinCat}_k^{\text{cpt-gen}}$ of k-linear, compactly generated, presentable, and stable ∞-categories as well as compact objects preserving, left adjoint functors, see [Coh13]. In particular, homotopy colimits in dgCat_k are described by ∞-categorical colimits in $\mathrm{LinCat}_k^{\text{cpt-gen}}$, see [Cis19, Rem. 7.9.10]. Note further that the forgetful functor $\mathrm{LinCat}_k^{\text{cpt-gen}} \to \mathrm{LinCat}_k$ preserves and reflects colimits. Thus, by Proposition 3.5, $\mathrm{Perf}(A * G)$ describes the homotopy colimit of ρ_A.

Proof of Proposition 3.5. By construction, there is a commutative diagram of compact objects preserving k-linear functors,

$$\begin{array}{ccc} & \mathcal{D}(A) & \\ {\scriptstyle F}\swarrow & & \searrow{\scriptstyle \mathcal{D}(F_A)} \\ \mathcal{D}(A)_G & \xrightarrow{\quad\zeta\quad} & \mathcal{D}(A * G) \end{array} \tag{3}$$

where F denotes the functor contained in the colimit diagram. Note that $\mathcal{D}(A)$ is generated under colimits by the objects arising from the objects in A, which are automatically compact. Similarly, the images of the objects in A under F and $\mathcal{D}(F_A)$ compactly generate, using that F is monadic by Lemma 2.11 and [Lur17, Prop. 4.7.3.14], respectively, that F_A is essentially surjective. It hence suffices to show that for all $X, Y \in A$, the morphism

$$\mathrm{Mor}_{\mathcal{D}(A)_G}(F(X), F(Y)) \to \mathrm{Mor}_{\mathcal{D}(A*G)}(\mathcal{D}(F_A)(X), \mathcal{D}(F_{\mathcal{C}})(Y)) \tag{4}$$

is an equivalence: given this, the fully faithfulness of the colimit preserving functor ζ on arbitrary objects follows from 'pulling the colimits out of the Homs', and the essential surjectivity of ζ follows from the compact generation by objects in A. We have equivalences

$$\mathrm{Mor}_{\mathcal{D}(A)_G}(F(X), F(Y)) \simeq \mathrm{Mor}_{\mathcal{D}(A)}(X, F^R F(Y)) \simeq \mathrm{Mor}_{\mathcal{D}(A)}(X, \coprod_{g \in G} g.Y) \simeq \coprod_{g \in G} \mathrm{Map}_A(g.X, Y)\,, \tag{5}$$

where the last equivalence uses that $X \in \mathcal{D}(A)$ is compact, and

$$\mathrm{Mor}_{\mathcal{D}(A*G)}(\mathcal{D}(F_A)(X), \mathcal{D}(F_A)(Y)) \simeq \mathrm{Map}_{A*G}(\tilde{X}, \tilde{Y}) = \coprod_{g \in G} \mathrm{Map}_A(g.X, Y)\,. \tag{6}$$

By the commutativity of the diagram (3), the morphism (4) restricts to an equivalence on the $\mathrm{Map}_A(X,Y)$-summands of (5) and (6). The morphism (4) also restricts to equivalences on the other summands which follows from combining the following two observations: Firstly, for every $g \in G$, the equivalence $F(g.X) \simeq F(X)$ is mapped to an equivalence $\widetilde{g.X} \simeq \mathcal{D}(F_A)(g.X) \simeq \tilde{X} = \mathcal{D}(F_A)(X)$. Secondly, together with the morphisms coming from A, these equivalences generate all morphisms in $\mathcal{D}(A)_G$ and $\mathcal{D}(A*G)$. □

Remark 3.7. In the case that A is an A_∞-category equipped with a strict group action as in [OZ22, AP24], the above construction easily adapts to show that the skew group A_∞-category $A*G$ of [OZ22, AP24] is equivalent to the ∞-categorical colimit over BG in the ∞-category $\mathrm{Cat}_{A_\infty}[M^{-1}]$ of A_∞-categories localized at Morita equivalences. For this, we use the equivalences of ∞-categories $\mathrm{Cat}_{A_\infty}[M^{-1}] \simeq \mathrm{dgCat}_k[M^{-1}] \simeq \mathrm{LinCat}_k^{\text{cpt-gen}}$, see [Pas24, COS24] and [Coh13].

Proof of Corollary 1.4. The derived ∞-categories $\mathcal{D}(A)$ and $\mathcal{D}(AG)$ of the algebras are equivalent to the module ∞-categories RMod_A and RMod_{AG}, respectively. Using Theorem 1.2 and Lemma 2.7, we thus have an equivalence of ∞-categories:

$$\mathcal{D}(AG) \simeq \lim_{BG} \mathcal{D}(A) = \mathcal{D}(A)^G\,.$$

By Remark 3.8, $N(\mathrm{Mod}(A)^G)$ embeds fully faithfully into $\mathcal{D}(A)^G$. The nerve $N(\mathrm{Mod}(AG))$ also embeds fully faithfully into $\mathcal{D}(AG)$ as the standard heart. It remains to note that these two full subcategories are identified under the above equivalence.

An object in $\mathcal{D}(AG)$ lies in the heart if and only if its image under the monadic functor $\mathrm{RHom}(AG,\text{-})\colon \mathcal{D}(AG) \to \mathcal{D}(A)$ lies in the heart of $\mathcal{D}(A)$. Under the equivalence with the limit $\mathcal{D}(A)^G$, this functor corresponds to the functor in the limit cone. An object in $\mathcal{D}(A)^G$ corresponds to a coCartesian section of the Grothendieck construction over BG, and lies in the image of $N(\mathrm{Mod}(A)^G)$ if and only if its evaluation at the unique object $* \in BG$ lies in $N(\mathrm{Mod}(A)) \subset \mathcal{D}(A)$. The evaluation functor at $* \in BG$ on coCartesian sections also describes the functor $\mathcal{D}(A)^G \to \mathcal{D}(A)$ in the limit cone. Hence, both full subcategories have the same essential images. □

Remark 3.8. We describe the relation of the 1-category of equivariant objects with limits in the ∞-category Cat_∞ of ∞-categories. Let Cat denote the 1-category of 1-categories. Consider the adjunction

$$h(\text{-})\colon \mathrm{Set}_\Delta \leftrightarrow \mathrm{Cat} \,:\! N(\text{-})$$

between the homotopy category functor and the simplicial nerve functor. Note that $N(\text{-})$ is fully faithful. This adjunction is a Quillen adjunction with respect to the Joyal model structure on Set_Δ and the standard model structure on Cat. We hence obtain an adjunction between ∞-categories $h(\text{-})\colon \mathrm{Cat}_\infty \leftrightarrow \mathrm{Cat}[W^{-1}] \,:\! N(\text{-})$, where W denotes the collection of equivalences of 1-categories. The derived functor $N(\text{-})$ is also fully faithful. Since it preserves ∞-categorical limits, and we see that limits in Cat_∞ of diagrams valued in nerves of 1-categories are equivalent to nerves of 1-categories, and hence determined by their homotopy categories. Given a diagram $\rho\colon BG \to \mathrm{Cat}$, $* \to C$, one finds that the homotopy category of the limit of $N \circ \rho\colon BG \to \mathrm{Cat}_\infty$ is equivalent to the category C^G of G-equivariant objects of C, which can be seen using the explicit description of limits via coCartesian sections of the Grothendieck construction [Lur24, Prop. 05RX]. Thus, we have

$$N(C^G) \simeq \lim_{BG} N \circ \rho \in \mathrm{Cat}[W^{-1}] \subset \mathrm{Cat}_\infty,.$$

3.2 Skew group dg-categories as homotopy colimits of cofibrant diagrams

In this section, we describe an alternative way to relate skew group dg-categories with group quotients using purely model categorical techniques.

We consider the functor category $\mathrm{Fun}(BG, \mathrm{dgCat}_k)$ as equipped with the projective model structure inherited from the quasi-equivalence model structure on dgCat_k. The weak equivalences are thus the pointwise quasi-equivalences and the fibrations are the pointwise fibrations.

Lemma 3.9. *Consider a functor $\rho_{\mathcal{C}}\colon BG \to \mathrm{dgCat}_k$ describing the action of a group G on a dg-category A. If A is cofibrant and the action free on the set of objects of A, then $\rho_A \in \mathrm{Fun}(BG, \mathrm{dgCat}_k)$ is a cofibrant object with respect to the projective model structure.*

Proof. Consider an acyclic fibration $\pi\colon B \to C$ and a morphism $F\colon A \to C$ in $\mathrm{Fun}(BG, \mathrm{dgCat}_k)$. To prove the cofibrancy of ρ_A, we must find a lift $A \to B$ of F along π in $\mathrm{Fun}(BG, \mathrm{dgCat}_k)$.

Since A is cofibrant in dgCat_k, we can choose a lift $\tilde{F}$ of F along π in dgCat_k. We can choose the lift $\tilde{F}$ which is G-equivariant on the level of objects: we fix an object $X \in A$ in each G-orbit of A. If for $g \in G$, we have $\tilde{F}(g.X) \neq g.\tilde{F}(X)$, we choose an equivalence $\tilde{F}(g.X) \simeq g.\tilde{F}(X)$ lifting the identity $F(g.X) = g.F(X)$ and redefine $\tilde{F}(g.X)$ as $g.\tilde{F}(X)$ and change the action on morphisms using this equivalence.

We choose a set R of representatives of the G-orbits of the objects of A. We define a new dg-functor $F'\colon A \to B$ as follows:

- On objects, F' is defined as $\tilde{F}$.
- For $X, Y \in R$ and $g_1, g_2 \in G$, we define $F'\colon\ \mathrm{Map}_A(g_1.X, g_2.Y) \to \mathrm{Map}_B(g_1.F'(X), g_2.F'(Y))$ on $\alpha \in \mathrm{Map}_A(g_1.X, g_2.Y)$ as $g_1.\tilde{F}(g_1^{-1}.\alpha)$.

We first note that F' defines a lift of F along π in dgCat_k: on objects that is clear and on a morphism $\alpha \in \mathrm{Map}_A(g_1.X, g_2.Y)$, we have

$$\pi \circ F'(\alpha) = \pi(g.\tilde{F}(g^{-1}.\alpha)) = g.\pi(\tilde{F}(g^{-1}.\alpha)) = g.F(g^{-1}.\alpha) = F(\alpha)$$

using the G-equivariance of F and π.

Finally, we check that F' is indeed G-equivariant, i.e. defines the desired lift in $\mathrm{Fun}(BG, \mathrm{dgCat}_k)$. On objects, this was noted above. Consider a morphism $\alpha \in \mathrm{Map}_A(g_1.X, g_2.Y)$ and $h \in G$. Then

$$F'(h.\alpha) = (hg_1).\tilde{F}((hg_1)^{-1}.(h.\alpha)) = h.(g_1.\tilde{F}(g_1^{-1}.\alpha)) = h.F'(\alpha)\,,$$

as desired. □

Proposition 3.10. *Under the assumptions of Lemma 3.9, the dg-category $(A * G)^{\mathrm{red}}$ from Definition 3.2 is the tip of a (strictly commuting) homotopy colimit cocone under the functor ρ_A with respect to the quasi-equivalence model structure on dgCat_k.*

Proof. It is straightforward to see that the cocone is a colimit cocone in the 1-category dgCat_k. Since the diagram $\rho_{\mathcal{C}}$ is cofibrant by Lemma 3.9, this colimit cocone is also a homotopy colimit cocone. □

4 Orbit categories

In this section, we will specialize to the case where $G = \mathbb{Z}$. The arising group quotients are known as orbit categories. After a general discussion in the ∞-categorical context in Section 4.1, we consider non-strict $\mathbb{Z}$-action on dg-categories in Section 4.2. Finally, we describe examples arising from periodic derived categories in Section 4.3.

4.1 General orbit ∞-categories

Let L denote the simplicial set with a unique 0-simplex $*$ and a unique non-degenerate 1-simplex $1\colon * \to *$.

Definition 4.1. Let $F\colon \mathcal{C} \to \mathcal{C}$ be a Mod_R-linear endofunctor of a Mod_R-linear ∞-category $\mathcal{C}$. The Mod_R-linear orbit ∞-category $\mathcal{C}/F$ is defined as the colimit of the functor

$$L \longrightarrow \mathrm{LinCat}_{\mathrm{Mod}_R}, \quad * \mapsto \mathcal{C}, \quad (1\colon * \to *) \mapsto F\,. \tag{7}$$

Remark 4.2. The orbit category $\mathcal{C}/F$ is also equivalent to the limit of the functor (7) and hence to the ∞-category of coCartesian sections of the Grothendieck construction of the functor (7). An object of $\mathcal{C}/F$ thus amounts to an object $X \in \mathcal{C}$ together with an equivalence $X \simeq F(X)$ in $\mathcal{C}$.

The lax limit of the functor (7), denoted $\mathcal{C}/^{\mathrm{lax}}F$, is given by the ∞-category of all sections of the Grothendieck construction of (7). Its objects are given by objects $X \in \mathcal{C}$ together with a morphism $X \to F(X)$ in $\mathcal{C}$.

Lemma 4.3. *Let $F\colon \mathcal{C} \to \mathcal{C}$ be a Mod_R-linear endofunctor.*

(1) The inclusion of simplicial sets $L \subset B\mathbb{N}, 1 \mapsto 1$ is inner anodyne. Pulling back along this inclusion hence gives rise to a commutative diagram of ∞-categories:

$$\begin{array}{ccc} \mathrm{Fun}(B\mathbb{N}, \mathrm{LinCat}_{\mathrm{Mod}_R}) & \xrightarrow{\simeq} & \mathrm{Fun}(L, \mathrm{LinCat}_{\mathrm{Mod}_R}) \\ & {\scriptstyle \mathrm{colim}} \searrow \quad \swarrow {\scriptstyle \mathrm{colim}} & \\ & \mathrm{LinCat}_{\mathrm{Mod}_R} & \end{array}$$

(2) Pulling back along the inclusion $B\mathbb{N} \subset B\mathbb{Z}$ yields a commutative diagram of ∞-categories,

$$\begin{array}{ccc} \mathrm{Fun}(B\mathbb{Z}, \mathrm{LinCat}_{\mathrm{Mod}_R}) & \hookrightarrow & \mathrm{Fun}(B\mathbb{N}, \mathrm{LinCat}_{\mathrm{Mod}_R}) \\ & {\scriptstyle \mathrm{colim}} \searrow \quad \swarrow {\scriptstyle \mathrm{colim}} & \\ & \mathrm{LinCat}_{\mathrm{Mod}_R} & \end{array}$$

where the horizontal functor is fully faithful. Its essential image consists of those functors $B\mathbb{N} \to \mathrm{LinCat}_{\mathrm{Mod}_R}$ mapping 1 to an equivalence.

In particular, the orbit category $\mathcal{C}/F$ of a Mod_R-linear equivalence $F\colon \mathcal{C} \to \mathcal{C}$ is equivalent to the colimit over a functor $B\mathbb{Z} \to \mathrm{LinCat}_{\mathrm{Mod}_R}$, mapping 1 to F.

Proof. We begin with showing part (1). That the inclusion $L \subset B\mathbb{N}$ is inner anodyne follows from applying [Lur09, 4.1.2.3] to the diagram

$$\begin{array}{ccccc} \Delta^1 & \longleftarrow & \{0,1\} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^{\mathbb{N}} & \longleftarrow & \mathbb{N} & \longrightarrow & * \end{array}$$

where the central $\mathbb{N}$ denotes the discrete simplicial set and $\Delta^{\mathbb{N}}$ denotes the nerve of the poset $\mathbb{N}$. The inclusion is thus final and cofinal by [Lur09, Prop. 4.1.1.3], showing the commutativity of the diagram in (1). The horizontal functor is an equivalence by [Lur24, Corollary 01EJ] and [Lur24, Proposition 01EF]

We next show part (2). The inclusion $B\mathbb{N} \subset B\mathbb{Z}$ is a weak homotopy equivalence and, using that $B\mathbb{Z}$ is a Kan complex, hence final and cofinal [Lur09, Cor. 4.1.2.6]. This shows the commutativity of the diagram in (2). The fully faithfulness of the horizontal functor amounts to the statement that $B\mathbb{Z}$ is the localization of $B\mathbb{N}$ at the morphism 1. This can be readily shown, for instance as follows. The localization of $B\mathbb{N}$ at 1 is equivalent to the localization of L at 1, since $L \subset B\mathbb{N}$ is a categorical equivalence. This localization can be obtained as the pushout of simplicial sets $L[\{1\}^{-1}] = L \amalg_{\Delta^1} Q$, where Q is a contractible Kan complex with 2 nondegenerate 0-simplices, see the proof of [Lur24, Proposition 01N4]. We observe that $L[\{1\}^{-1}]$ is a Kan complex and equivalent to $B\mathbb{Z}$, since the geometric realizations of both are equivalent to the circle S^1. $\square$

4.2 Orbit dg-categories

Given a dg-category A, we denote by $\mathrm{rep}_{\mathrm{dg}}(A,A)$ the dg-category of cofibrant dg A-A-bimodules which are right quasi-representable, see also [Kel06, FKQ24] for detailed definitions. We note that any such bimodule $F \in \mathrm{rep}_{\mathrm{dg}}(A,A)$ gives rise to a dg-functor $\mathrm{Perf}(F)\colon \mathrm{Perf}(A) \to \mathrm{Perf}(A)$, and thus to a Mod_k-linear functor $\mathcal{D}(F)\colon \mathcal{D}(A) \to \mathcal{D}(A)$ between the derived ∞-categories. We call $F \in \mathrm{rep}_{\mathrm{dg}}(A,A)$ a Morita equivalence if the dg-functor $\mathrm{Perf}(F)$ is a quasi-equivalence (or equivalently $\mathcal{D}(F)$ is an equivalence of ∞-categories).

Given $F \in \mathrm{rep}_{\mathrm{dg}}(A,A)$, [FKQ24] defines the dg-orbit category $A/F^{\mathbb{Z}}$ as a dg-localization of the 'left lax quotient dg-category' $A/_{ll}F^{\mathbb{N}}$, which is the dg-category with the same objects as A and morphism complexes $\mathrm{Map}_{A/_{ll}F^{\mathbb{N}}}(X,Y) = \bigoplus_{i\in\mathbb{N}} \mathrm{Map}_A(X, F^i(Y))$.

Remark 4.4. Supposing that $F \in \mathrm{rep}_{\mathrm{dg}}(A,A)$ is a Morita equivalence, one can show that for all $X,Y \in A$, there is a quasi-isomorphism

$$\mathrm{Map}_{A/F^{\mathbb{Z}}}(X,Y) \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{Map}_{\mathrm{Perf}(A)}(X, F^i(Y))\,. \tag{8}$$

Proposition 4.5. *Let $F \in \mathrm{rep}_{\mathrm{dg}}(A,\mathcal{A})$ be a Morita equivalence. Then there exists a canonical equivalence of Mod_k-linear ∞-categories*

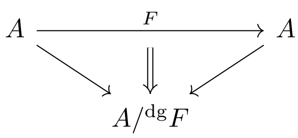
$$\mathcal{D}(A)/\mathcal{D}(F) \simeq \mathcal{D}(A/F^{\mathbb{Z}})$$

between the Mod_k-linear orbit ∞-category and the derived ∞-category of the dg-orbit category.

Remark 4.6. We expect that $\mathcal{D}(A)/^{\mathrm{lax}}\mathcal{D}(F)$ is equivalent to the derived ∞-category of the left lax quotient dg-category $A/_{ll}F^{\mathbb{N}}$. We further expect Proposition 4.5 to also hold true if F is not a Morita equivalence.

Proof of Proposition 4.5. As shown in [FKQ24], there exists a dg-functor $\pi\colon A \to A/F^{\mathbb{Z}}$ together with a natural equivalence $\pi \circ F \simeq F$. Passing to compact, cofibrant modules, F induces a dg-functor $\mathrm{Perf}(F)\colon \mathrm{Perf}(A) \to \mathrm{Perf}(A)$ (and not just a dg-bimodule) together with a natural equivalence $\mathrm{Perf}(\pi) \circ \mathrm{Perf}(F) \simeq \mathrm{Perf}(F)$. This exhibits $A/F^{\mathbb{Z}}$ as the tip of a cocone under the functor $L \to \mathrm{dgCat}_k[M^{-1}]$, $1 \mapsto \mathrm{Perf}(F)$. Passing to derived ∞-categories, this induces a Mod_k-linear functor $\mathcal{D}(A)/\mathcal{D}(F) \to \mathcal{D}(A/F^{\mathbb{Z}})$, which is checked to be an equivalence using an analogous argument as in the proof of Proposition 3.5. □

Remark 4.7. Suppose that $F\colon A \to A$ is a dg-functor. An a priori different definition of orbit dg-category is given in [Kel05], we will denote this orbit category by $A/^{\mathrm{dg}}F$. There is an apparent diagram of dg-categories,



where the natural transformation evaluates at $X \in A$ to the canonical morphism $F(X) \to X$ in $A/^{\mathrm{dg}}F$. Using the universal property shown in [FKQ24], this induces a dg-functor $A/_{ll}F^{\mathbb{N}} \to A/^{\mathrm{dg}}F$, which in turn induces a dg-functor $\alpha\colon A/F^{\mathbb{Z}} \to A/^{\mathrm{dg}}F$. If F is a Morita equivalence, the quasi-isomorphisms (8) imply that α is a a quasi-equivalence.

In the case that F is strictly invertible, the orbit dg-category $A/^{\mathrm{dg}}F$ is furthermore isomorphic to the skew group dg-category $A * \mathbb{Z}$ from Section 3.

4.3 Example: Periodic derived categories

Let $\mathrm{Perf}(k)$ be the dg-category of finite dimensional chain complexes over a field k. Consider the invertible endofunctor $[n]\colon \mathrm{Perf}(k) \to \mathrm{Perf}(k)$ given by the shift functor. This defines a $\mathbb{Z}$-action on $\mathrm{Perf}(k)$, with orbit dg-category $\mathrm{Perf}(k)/[n]$.

Let $k[t_n]$ be the dg-algebra of graded polynomials with the monomial t_n in degree n (in homological grading). Note that $k[t_n]$ is the $(n+1)$-Calabi–Yau completion of k in the sense of [Kel11]. We similarly denote by $k[t_n^{\pm}]$ the dg-algebra of graded Laurent polynomials.

Proposition 4.8. *There are equivalences in* $\mathrm{LinCat}_{\mathrm{Mod}_k}$

$$\mathcal{D}(\mathrm{Perf}(k)/[n]) \simeq \mathcal{D}(k)/[n] \simeq \mathcal{D}(k[t_n^{\pm}])\,.$$

Proof. The first equivalence is the statement of Proposition 4.5. For the second equivalence, we first produce a functor $\mathcal{D}(k)/[n] \to \mathcal{D}(k[t_n^{\pm}])$ using the universal property of the colimit. We employ the following trick, to avoid any discussions about signs for the shift functor: any Mod_k-linear functor $D(k) \to \mathcal{D}(k[t_n^{\pm}])$ is fully determined by the image of k. Since $k[t_n^{\pm}][n] \simeq k[t_n^{\pm}] \in \mathcal{D}(k[t_n^{\pm}])$, we thus find a commutative diagram in $\mathrm{LinCat}_{\mathrm{Mod}_k}$:

$$\begin{array}{ccc} \mathcal{D}(k) & \xrightarrow{\quad [n] \quad} & \mathcal{D}(k) \\ & {\scriptstyle k\mapsto k[t_n^{\pm}]} \searrow \quad \swarrow {\scriptstyle k\mapsto k[t_n^{\pm}][n]} & \\ & \mathcal{D}(k[t_n^{\pm}]) & \end{array}$$

This induces a functor $\mathcal{D}(k)/[n] \to \mathcal{D}(k[t_n^{\pm}])$, which is now readily checked to be fully faithful and essentially surjective. □

Remark 4.9. Let $\mathcal{D}^{\mathrm{fin}}(k[t_n]) \subset \mathcal{D}^{\mathrm{perf}}(k[t_n])$ denote the full subcategory of modules with finite dimensional total homology. There is an equivalence in $\mathrm{LinCat}_{\mathrm{Mod}_k}$

$$\mathcal{D}(k[t_n])/\operatorname{Ind}\mathcal{D}^{\mathrm{fin}}(k[t_n]) \simeq \mathcal{D}(k[t_n^{\pm}])\,,$$

see [Chr22, Lem. 2.11]. By [HI22, Rem. 2.14.(2)], there is an equivalence of dg-categories

$$\mathrm{Perf}(k)/[n] \simeq \mathrm{Perf}(k[t_n])/\mathrm{Perf}(k[t_n])^{\mathrm{fin}}\,,$$

where $\mathrm{Perf}(k[t_n])^{\mathrm{fin}}$ denotes the dg-category subcategory of $\mathrm{Perf}(k[t_n])$ of dg-modules with finite dimensional total homology. Combining these equivalences provides an alternative way to prove Proposition 4.8

Lemma 4.10. *Let $\mathcal{C}$ be a Mod_k-linear ∞-category. Then the orbit category $\mathcal{C}/[n]$ is equivalent to the tensor product $\mathcal{C} \otimes_{\mathrm{Mod}_k} \mathcal{D}(k[t_n^{\pm}])$ in $\mathrm{LinCat}_{\mathrm{Mod}_k}$.*

Proof. Using that $\mathcal{D}(k) \simeq \mathrm{Mod}_k$ and that the tensor product in $\mathrm{LinCat}_{\mathrm{Mod}_k}$ preserves colimits in the second entry, we find equivalences

$$\mathcal{C} \otimes_{\mathrm{Mod}_k} \mathcal{D}(k[t_n^{\pm}]) \simeq \mathcal{C} \otimes_{\mathrm{Mod}_k} \operatorname{colim}_L \mathrm{Mod}_k \simeq \operatorname{colim}_L \mathcal{C} = \mathcal{C}/[n]\,.$$

□

The above allows us to obtain the following concrete description of the 1-periodic topological Fukaya category, or equivalently Higgs category, of a marked surface studied in [Chr22].

Proposition 4.11. *Let A be a finite dimensional gentle algebra and* **S** *the corresponding marked surface in the sense of [OPS18]. Assume that every boundary component of* **S** *contains at least one marked point*[3]*. Then the Higgs category $\mathcal{H}$ [Wu23] associated with* **S**, *described in [Chr22], is equivalent to the perfect derived ∞-category of the orbit dg-category* $\mathrm{Perf}(A)/[1]$.

In particular, the 1-categorical orbit category $(\operatorname{ho}\mathrm{Perf}(A))/[1]$ *embeds fully faithfully into the triangulated homotopy 1-category* $\operatorname{ho}\mathcal{H}$ *of the Higgs category.*

[3] This is equivalent to the assertion that A is smooth, which follows from [LP20, Lem. 3.1.3].

Proof of Proposition 4.11. It is shown in [Chr22] the Ind-completion of the ∞-categorical Higgs category $\mathcal{H}$ is equivalent to $\mathcal{D}(A)\otimes_{\mathrm{Mod}_k}\mathcal{D}(k[t_1^{\pm}])$, see Proposition 4.16 and Theorem 8.4 in loc. cit.. The statement thus reduces to Lemma 4.10. ☐

5 Spectral skew group algebras

We fix a base $\mathbb{E}_\infty$-ring spectrum R. Consider a Mod_R-linear ∞-category $\mathcal{C}$ with a G-action. In the case that $\mathcal{C}\simeq \mathrm{RMod}_A$ has a compact generator A, the functor $\mathrm{Mor}_{\mathcal{C}}(A,\text{-})\colon \mathcal{C}\to \mathrm{Mod}_R$ is monadic, and hence so is the composite $\mathcal{C}_G\to\mathcal{C}\to\mathrm{Mod}_R$. The image of R under the left adjoint adjoint $\mathrm{Mod}_R\to\mathcal{C}_G$ defines a compact generator Y of $\mathcal{C}_G$. In the case that A is fixed by the G-action, the R-linear endomorphism algebra of Y can be considered the corresponding skew group algebra AG. We study this situation in more detail in the following.

We begin with a more systematic construction of the skew group algebra AG.

Construction 5.1. Let $\underline{\mathrm{Mod}_R}\in \mathrm{Fun}(BG,\mathrm{LinCat}_{\mathrm{Mod}_R})$ denote the constant functor with value Mod_R. We define the functor ξ as the following composite:

$$\begin{aligned}\mathrm{Fun}(BG,(\mathrm{LinCat}_{\mathrm{Mod}_R})_{\mathrm{Mod}_R/}) &\to \mathrm{Fun}(BG,\mathrm{LinCat}_{\mathrm{Mod}_R})_{\underline{\mathrm{Mod}_R}/}\\ &\to \mathrm{Fun}(BG,\mathrm{LinCat}_{\mathrm{Mod}_R})_{\mathrm{Mod}_R^{\amalg G}/}\\ &\simeq \left(\mathrm{LinCat}_{\mathrm{Mod}_R^{\amalg G}}\right)_{\mathrm{Mod}_R^{\amalg G}/}.\end{aligned}$$

The G action on $\mathrm{Mod}_R^{\amalg G}$ is given by permuting the G-components, so that $\mathrm{colim}_{BG}\,\mathrm{Mod}_R^{\amalg G}\simeq \mathrm{Mod}_R\in\mathrm{LinCat}_{\mathrm{Mod}_R}$. This equivalence is adjoint to a morphism $\mathrm{Mod}_R^{\amalg G}\to\underline{\mathrm{Mod}_R}$ in $\mathrm{Fun}(BG,\mathrm{LinCat}_{\mathrm{Mod}_R})$ which is used for the second functor above. The third functor uses the equivalence from Proposition 2.8.

Proposition 5.2. *There is a commutative diagram of ∞-categories*

$$\begin{array}{ccccc}\mathrm{Fun}(BG,\mathrm{Alg}(\mathrm{Mod}_R)) & \xhookrightarrow{\mathrm{Fun}(BG,\Theta_*)} & \mathrm{Fun}(BG,(\mathrm{LinCat}_{\mathrm{Mod}_R})_{\mathrm{Mod}_R/}) & \longrightarrow & \mathrm{Fun}(BG,\mathrm{LinCat}_{\mathrm{Mod}_R})\\ \downarrow & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\simeq}\\ \mathrm{Alg}(\mathrm{Mod}_R^{\amalg G}) & \xhookrightarrow{\Theta_*} & \left(\mathrm{LinCat}_{\mathrm{Mod}_R^{\amalg G}}\right)_{\mathrm{Mod}_R^{\amalg G}/} & \longrightarrow & \mathrm{LinCat}_{\mathrm{Mod}_R^{\amalg G}}\end{array}$$

where the right vertical functor is the equivalence from Proposition 2.8. We denote the left vertical functor by

$$\xi^{\mathrm{Alg}}\colon \mathrm{Fun}(BG,\mathrm{Alg}(\mathrm{Mod}_R))\to\mathrm{Alg}(\mathrm{Mod}_R^{\amalg G})\,.$$

Proof. The commutativity of the right diagram follows from the definition of ξ. To show the commutativity of the left square, we need to prove that elements in the image of $\mathrm{Fun}(BG,\Theta_*)$ are mapped by ξ to elements in the image of Θ_*.

Using [Lur17, Prop. 4.8.5.8], we find that a functor $BG\to(\mathrm{LinCat}_{\mathrm{Mod}_R})_{\mathrm{Mod}_R/}, *\mapsto(\mathrm{Mod}_R\to\mathcal{C})$ factors through $\mathrm{Alg}(\mathrm{Mod}_R)$ if and only if the image of $R\in\mathrm{Mod}_R$ in $\mathcal{C}$, denoted $Y\in\mathcal{C}$, is a compact generator preserved by the G-action. Note that condition (6) in Proposition 4.8.5.8 is automatically fulfilled, since Mod_R is generated by R, compare also with the proof of [Lur17, Thm. 7.1.2.1].

Let $\mathrm{Mod}_R\to\mathcal{C}$, $R\mapsto Y$ be as before, with Y a compact generator. We denote $A=\mathrm{End}_{\mathcal{C}}(Y)\in\mathrm{Alg}(\mathrm{Mod}_R)$, and have $\mathcal{C}\simeq\mathrm{RMod}_A$. The functor $\mathrm{Mod}_R\to\mathcal{C}\simeq\mathrm{RMod}_A$ is given by $(\text{-})\otimes A$. Its image under ξ is equivalent to the functor $F\colon\mathrm{Mod}_R^{\amalg G}\to\mathrm{RMod}_A$, componentwise given by $(\text{-})\otimes A$, i.e. satisfying $F(R_g)\simeq A$ for all $g\in G$.

We record here the curious fact that the functors F and F^R only depend on the Mod_R-linear structure of $\mathcal{C}$, and not on the $\mathrm{Mod}_R^{\times G}$-linear structure, meaning the G-action on $\mathcal{C}$. This is because A is fixed by the G-action on $\mathcal{C}$. The G-action will only contribute when describing the $\mathrm{Mod}_R^{\amalg G}$-linear endomorphism algebra structure of $F^R(A)$, see Remark 5.4 below.

We again apply [Lur17, Prop. 4.8.5.8] to show that the functor $F : \mathrm{Mod}_R^{\amalg G} \to \mathrm{RMod}_A$ lies in the image of Θ_*, thus showing the existence of the left commutative square. Conditions (1), (2), (3) and (5) are clear. For condition (4), it suffices to observe that the right adjoint F^R of $F\colon \mathrm{Mod}_R^{\amalg G} \to \mathrm{RMod}_A$ is G-componentwise a functor that preserves geometric realizations, and thus preserves geometric realizations as well. Condition (6) boils down to the statement that F^R and the counit of $F \dashv F^R$ commute with the $\mathrm{Mod}_R^{\amalg G}$-action, which is straightforward to check. □

Definition 5.3. Consider an action $\rho\colon BG \to \mathrm{Alg}(\mathrm{Mod}_R)$ of the group G on an R-linear ring spectrum A. The skew group algebra $AG \in \mathrm{Alg}(\mathrm{Mod}_R)$ is defined as the image of ρ under the functors

$$\mathrm{Fun}(BG, \mathrm{Alg}(\mathrm{Mod}_R)) \xrightarrow{\xi^{\mathrm{Alg}}} \mathrm{Alg}(\mathrm{Mod}_R^{\amalg G}) \xrightarrow{\mathrm{Alg}(\psi)} \mathrm{Alg}(\mathrm{Mod}_R) \tag{9}$$

from Remark 2.3 and Proposition 5.2.

Remark 5.4. We unravel the construction of the skew group algebra AG.

Starting with an R-linear ring spectrum A, with the group G acting on it, we have an induced G-action on its ∞-category of modules RMod_A, which we can also consider as a left-tensoring of RMod_A by $\mathrm{Mod}_R^{\amalg G}$. Considering A as an object of RMod_A, we consider the $\mathrm{Mod}_R^{\amalg G}$-linear endomorphism algebra $\tilde{A} \in \mathrm{Alg}(\mathrm{Mod}_R^{\amalg G})$ of A, in the sense of [Lur17, Section 4.7.1]. The algebra $\tilde{A}$ is a refinement of the skew group algebra AG, which is defined as the image of $\tilde{A}$ under the monoidal functor $\psi = \coprod\colon \mathrm{Mod}_R^{\amalg G} \to \mathrm{Mod}_R$. It remains to unravel its definition to describe the skew group algebra more explicitly.

Given $g \in G$, the restriction to the corresponding component $\mathrm{Mod}_R \subset \mathrm{Mod}_R^{\amalg G}$ of $\tilde{A}$ is given by A. Inspecting the definition, one sees that this amounts to the fact that A is preserved by the action of g. We write this decomposition as $\tilde{A} = \coprod_{g\in G} \tilde{A}_g$. Note that we thus have $AG \simeq \coprod_{g\in G} A$.

We turn to the algebra structure of $\tilde{A}$ (describing it on the level of homotopy groups). To make the notation for this more transparent, we denote the $\mathrm{Mod}_R^{\amalg G}$-action by $\otimes^G$. Firstly, we consider $\tilde{A}$ as equipped with the map $m\colon \tilde{A} \otimes^G A \to A$ in RMod_A, that is simply the multiplication map of A on every component $A \otimes A \simeq \tilde{A}_g \otimes^G A$. The crucial point of the construction of $\tilde{A}$ in [Lur17, Section 4.7.1] is that the multiplication map $\tilde{m}\colon \tilde{A} \otimes^G \tilde{A} \to \tilde{A}$ in $\mathrm{Mod}_R^{\amalg G}$ is uniquely determined by the property that the following diagram commutes[4]:

$$\begin{array}{ccc} \tilde{A} \otimes^G \tilde{A} \otimes^G A & \xrightarrow{\tilde{A}\otimes^G m} & \tilde{A} \otimes^G A \\ \Big\downarrow{\scriptstyle \tilde{m}\otimes^G A} & & \Big\downarrow{\scriptstyle m} \\ \tilde{A} \otimes^G A & \xrightarrow{\quad m \quad} & A \end{array}$$

We write elements of the homotopy groups of $\tilde{A}$ as pairs (g, a) with $g \in G$ and $a \in A$. We next argue that the morphism

$$\tilde{A} \otimes^G m\colon \tilde{A}_g \otimes^G \tilde{A} \otimes^G A \to \tilde{A}_g \otimes^G A$$

is given on elements by

$$(g, a) \otimes (h, b) \otimes c \mapsto (g, a) \otimes ((g.b) \cdot c) \ .$$

The above identity follows from the following three observations:

- Every element $(h, b) \in \tilde{A}$ determines an inclusion $A \hookrightarrow \tilde{A} \otimes^G A$, mapping c to $(h, b) \otimes c$, in RMod_A.

[4] This follows from the fact that $m\colon \tilde{A} \otimes^G A \to A$ is a terminal object in the monoidal ∞-category $\mathrm{Mod}_R^{\amalg G}[A]$ described in [Lur17, Section 4.7.1].

- For every $(h, b) \in \tilde{A}$, the following diagram commutes:

$$\begin{array}{ccc} A & & \\ {\scriptstyle c\mapsto(h,b)\otimes c}\downarrow & \overset{b\cdot(\text{-})}{\searrow} & \\ \tilde{A} \otimes^G A & \xrightarrow{m} & A \end{array}$$

- For every $b \in A$ the tensor product $R_g \otimes^G (\text{-})$ with the object $R_g = R$ lying in the g-component of $\mathrm{Mod}_R^{\amalg G}$ maps the endomorphism $A \xrightarrow{b\cdot(\text{-})} A$ to the endomorphism $A \xrightarrow{g.b\cdot(\text{-})} A$, where $g.$ refers to the G-action on A.

With the above, it is now straightforward to see that $\tilde{m}$ is given by

$$\tilde{m} \colon \tilde{A} \otimes \tilde{A} \to \tilde{A}, \quad (g, a) \otimes (h, b) \mapsto (gh, a \cdot (g.b)),$$

matching the desired formula of the multiplication of the skew group algebra.

In case that A is a discrete algebra object of Mod_k for a field k, the skew group algebra AG is thus isomorphic to the classical skew group algebra. If A is a dg-algebra with a strict G-action, one can deduce from Theorem 1.2 and Proposition 3.5 that the dg-categorical skew group algebra $A * G$ is quasi-isomorphic to AG.

Finally, we show that the module ∞-category of AG describes the group quotient.

Proof of Theorem 1.2. Let $\tilde{A} \in \mathrm{Alg}(\mathrm{Mod}_R^{\amalg G})$ be the image of A under the functor ξ^{Alg}. We have equivalences

$$\mathrm{colim}_{BG}\, \mathrm{RMod}_A \simeq \psi_!(\mathrm{RMod}_A) \simeq \psi_!(\mathrm{RMod}_{\tilde{A}}(\mathrm{Mod}_R^{\amalg G})) \simeq \mathrm{RMod}_{\psi(\tilde{A})} = \mathrm{RMod}_{AG},$$

where the first equivalence follows from Lemma 2.10, the second from the commutativity of the left square in Proposition 5.2, and the third (essentially) from [Lur17, Thm. 4.8.4.6]. ☐

Remark 5.5. Contrary to the results of the previous sections, the construction of the skew group algebra and the proof of Theorem 1.2 do not use that every element of G has an inverse. Theorem 1.2 thus immediately generalizes to the case that G is a monoid in sets. We also note that a small amount of additional work allows a generalization to the setting where G is a monoid in spaces.

Example 5.6. The skew group algebra typically does not describe the colimit over BG in the ∞-category of algebra objects in Mod_R. Sometimes, they are however Morita equivalent.

Consider for instance the case that k is a field, $A = k^{\oplus 2}$ is the product algebra and $G = \mathbb{Z}/2\mathbb{Z}$, acting on A by permuting the two factors. The colimit $\mathrm{colim}_{B\mathbb{Z}/2\mathbb{Z}} A \in \mathrm{Alg}(\mathrm{Mod}_k)$ is then equivalent to k if $\mathrm{char}(k) \neq 2$ and equivalent to 0 if $\mathrm{char}(k) = 2$. The skew group algebra AG is isomorphic to the matrix algebra of 2×2-matrices, which is Morita equivalent to k.

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